

Brane tilings and M2 branes

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Brane tilings and M2 branes

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ABSTRACT: Brane tilings are efficient mnemonics for Lagrangians of $\mathcal{N} = 2$ Chern-Simons-matter theories. Such theories are conjectured to arise on M2-branes probing singular toric Calabi-Yau fourfolds. In this paper, a simple modification of the Kasteleyn technique is described which is conjectured to compute the three dimensional toric diagram of the non-compact moduli space of a single probe. The Hilbert Series is used to compute the spectrum of non-trivial scaling dimensions for a selected set of examples.

KEYWORDS: Chern-Simons Theories, M-Theory, AdS-CFT Correspondence

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1 Introduction

The recent work on 2+1 dimensional superconformal Chern-Simons theories which started with the discovery of the BGL $\mathcal{N} = 8$ supersymmetric Chern-Simons theory [1–7] and culminated in the construction of the ABJM superconformal $\mathcal{N} = 6$ Chern-Simons theory [8],

has shed new light on the the AdS_4/CFT_3 correspondence. A long-standing problem in 2+1 dimensional superconformal theories is the identification of the gauge theory dual to an $AdS_4 \times H$ supersymmetric background. While this problem is relatively well understood in 3+1 dimensions, it is much less understood in 2+1 dimensions. In the past, attempts to find duals have focused on Yang-Mills theories flowing in the IR to superconformal fixed points [9–12]. It seems now that supersymmetric Chern-Simons theories can do a better job. The $\mathcal{N} = 6$ ABJM theory nicely incorporates all relevant features of a dual theory for the backgrounds with large supersymmetry, including the maximally supersymmetric case of $H = S^7$. Other examples of superconformal Chern-Simons theories with supersymmetry $\mathcal{N} = 3, 4, 5$ have been constructed recently [13–20].

It is known that $AdS_4 \times H$ M-theory backgrounds, with H a seven dimensional Sasaki-Einstein manifold, preserve $\mathcal{N} = 2$ supersymmetry¹ [21–23]. The $X = C(H)$ cone over H is a Calabi-Yau four-fold and the backgrounds of interest arise as near-horizon geometries of membranes sitting at the singular tip of the cone. We thus have a correspondence between the infinite number of Calabi-Yau four-folds and an infinite set of superconformal theories. The open problem is to find the explicit correspondence. The analogous problem in 3+1 dimensions has been solved, at least for the class of toric Calabi-Yau singularities, using Brane Tilings [24–26]. A similar proposal for 2+1 dimensions is based on crystals [27–29] but it is not as well understood as in 3+1 dimensions. Various recent progresses have been made using Chern-Simons theories.

A 2+1 dimensional theory dual to an AdS_4 M-background should have various distinctive features. In particular, the abelian moduli space should be a four-dimensional Calabi-Yau cone X and the non-abelian moduli space should be the symmetrized product of N copies of X (or a modification of it). In [30], we showed how to construct infinitely many Chern-Simons theories with these properties using tilings. For every periodic tiling of the torus that gives rise to a consistent 3+1 dimensional superconformal gauge theory, and for every choice of Chern-Simons parameters we constructed a Chern-Simons theory with a component of the moduli space which is the symmetric product of a Calabi-Yau four-fold. Each tiling gives rise therefore to an infinite family of Calabi-Yau four-folds depending on the integer Chern-Simons parameters. As independently noticed by various authors [19, 30, 31], the abelian moduli space of quiver Chern-Simons theory is naturally four-dimensional. In [30] we demonstrated that the mesonic abelian moduli space of 2+1 dimensional quivers arising from tilings is always a toric Calabi-Yau cone and that the full mesonic moduli space is generically a symmetric product.²

In this paper we continue our analysis of the class of Chern-Simons theories obtained from tilings following two important directions.

First of all, it is important to find an efficient forward algorithm, i.e. a prescription for determining the toric diagram of the Calabi-Yau four-fold. This problem was addressed in [30] by explicitly looking at the supersymmetric moduli space of vacua as a set of solution of F and D terms. The recent advances in understanding the space of solutions of the F-

¹In 2+1 dimensions we can also have $\mathcal{N} = 1$ supersymmetry but we do not consider this case.

²See [31] for an independent analysis.

terms for a toric quiver [32, 33] - the *master space* as we dubbed it - allows to work out easily many examples from this perspective [30]. This forward algorithm, however, become increasingly cumbersome as the size of the tiling is increased. We need therefore to find a fast forward algorithm, similar to that existing in 3+1 dimensions [25]. We indeed show that, a mild modification of the 3+1 dimensional algorithm gives an efficient fast forward algorithm also in 2+1 dimensions: it is based, as in 3+1 dimensions, on the concepts of perfect matchings and of the Kasteleyn matrix.

Secondly, it is important to study the quantum properties of the Chern-Simons theory and its spectrum of conformal dimensions from the dual supergravity perspective. The Chern-Simons theories are expected to flow to IR fixed points. The superpotential is not always quartic and therefore the R-charge of the fields and the dimension of gauge invariant operators are quantum corrected. There are many abelian global symmetries in the quiver that mix with the R-symmetry. There is a notion of exact R-symmetry at the IR fixed point, which is the one sitting in the superconformal algebra, but it is extremely difficult to find it. At the moment of this writing, the many efficient 3+1 dimensional tools for studying superconformal theories are not available in 2+1 dimensions. In particular we have nothing similar to a-maximization [34] to predict the exact R-symmetry. However, the spectrum of conformal dimensions of a Chern-Simons theory dual to an $AdS_4 \times H$ background can be predicted from supergravity. We use geometrical methods to find the exact R-charges of the dual Chern-Simons theory — in particular minimization of the volume functional and computation of volumes [35]. In 3+1 dimensions these tools are the geometrical counterpart of a-maximization [36–38]. We show, in all our examples, that the supergravity results for the mesonic spectrum are consistent with the Chern-Simons expectations and extremely similar in structure to the results in 3+1 dimensions. We also look at baryons that, as familiar in the AdS/CFT correspondence, correspond to wrapped branes and we point out some puzzles. We notice that the consistency of the duality requires the understanding of the absence of certain supersymmetric states in the gravitational dual or, equivalently, the presence of new states in the Chern-Simons theory.

It is important to know what are the prediction of supergravity. Although we cannot compare the result with exact field theory computations, we can predict the full spectrum of dimensions of the dual Chern-Simons theory and we can make various checks of the overall consistency of the construction. It is not obvious for example that every tiling gives rise to a consistent superconformal theory. Moreover, it seems that various different tilings, and more generally various different quivers, give rise to the same Calabi-Yau four-fold. The knowledge of the spectrum may help in understanding whether these models are equivalent or which is the best candidate for a duality.

We would like to stress the important role played by the master space of the 3+1 dimensional model [32]. The master space encodes all the properties of the Chern-Simons theory in a beautiful way. In particular, hidden symmetries in the master space of some tilings [32], that are still mysterious from the point of view of 3+1 dimensions, reveal their role in the 2+1 dimensional theories.

The paper is organized as follows. In section 2 we review the construction of Chern-Simons theories from tilings [30]. In section 3 we explain the fast forward algorithm in

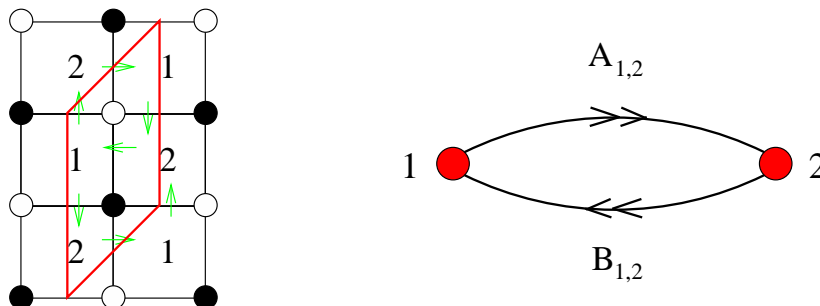


Figure 1. (i) Brane tiling for \mathbb{C}^4 (and $\mathbb{C}^4/\mathbb{Z}_k$). The fundamental domain is shown in red. The green arrows indicate the direction of the bifundamental fields based on the convention that the black node is on the left-hand side. (ii) The corresponding quiver.

terms of the Kasteleyn matrix and the procedure for determining the toric data of the four-dimensional Calabi-Yau singularity. In section 4 we discuss the physical meaning of the Hilbert series. We also distinguish between mesonic and baryonic spectra. In section 5 we elaborate on many examples. The paper ends with conclusions and comments. The first appendix contains technical results on the computation of Hilbert series that are used in the main text. The second appendix describes some observations about brane crystals and some speculations about a possible (Seiberg) dual theory to the ABJM theory at level one.

While finishing this work, two papers appeared that have partial overlap with section 3 [39, 40] and that present an algorithm for computing the toric data of the Calabi-Yau four-fold. These algorithms seem to be equivalent to the fast forward algorithm presented here.

2 Brane tilings and M2 branes

Recall from [30] that for every (consistent) periodic, bipartite, two dimensional tiling of the plane we can construct a $\mathcal{N} = 2$ (4 supercharges) CS theory in 2+1 dimensions whose abelian moduli space is a toric Calabi-Yau four-fold.

The rules for writing down the 2+1 dimensional theory follow the rules set out for the 3+1 dimensional theory [24, 25]. Every face is a $U(N)$ gauge group and every edge is a chiral superfield transforming in a bifundamental representation of the two gauge groups it separates with orientation defined by the bipartite structure of the tiling. By convention one can pick an (incoming) outgoing arrow to correspond to an (anti)-fundamental representation, respectively. Every vertex in the tiling contributes a term in the superpotential given by the products of all the fields that meet at the vertex,³ with a positive sign for white vertices and a negative sign for black ones. This set of rules corresponds to a Hanany-Witten construction of the theory, where the faces of the tiling are D4-branes bounded by NS-branes and chiral superfields arise from open strings connecting adjacent faces. As the theory flows to the IR, one should lift the configuration to M-theory where it becomes a theory of M2 branes.

An example (in fact the simplest) is the ABJM theory with $\mathcal{N} = 6$ supersymmetry [8]. It is obtained from the tiling that gives rise to the conifold theory in 3+1 dimensions. This

³The coefficients in the superpotential are not encoded in the tilings.

is shown in figure 1(i) with the quiver in figure 1(ii). There are two gauge groups, fields $A_i, B_i, i = 1, 2$ transforming in the (N, \bar{N}) and (\bar{N}, N) representation of the gauge group, respectively, and interacting with the superpotential

$$W = A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1. \tag{2.1}$$

In 3+1 dimensions, we would introduce standard kinetic terms for every gauge group and we would obtain a superconformal quiver gauge theory with an abelian moduli space that is a toric Calabi-Yau three-fold.

In 2+1 dimensions, we are not introducing kinetic terms for the gauge fields but instead CS interactions. For each edge E_i we add an $\mathcal{N} = 2$ preserving Chern-Simons interaction with the following rule: add an integer CS coefficient k_i and $-k_i$ to the adjacent gauge groups connected by the edge. Call k_a the resulting CS coefficient for the a -th gauge group. The supersymmetric vacua are obtained as usual as solutions of the F- and D-term constraints. Consider for the moment the abelian theory. As extensively discussed in [30], the F-term constraints are given by the vanishing of all derivatives of the superpotential, as in 3+1 dimensions, and the D-terms constraints can be summarized by the following equations

$$\mu_a(X) = 4k_a \sigma \tag{2.2}$$

where $\mu_a(X)$ is the moment map for the action of the a -th group (the 3+1 dimensional D-term) and σ is an auxiliary field in the vector supermultiplets. Since $\sum_a k_a = 0$ and $\sum_a \mu_a(X) = 0$ by construction, one of these equations is redundant; the overall U(1) does not enter in the supersymmetric vacua conditions, as usual. Moreover, any other linear combination of gauge groups with coefficient m_a orthogonal to the CS parameters $\sum_a k_a m_a = 0$ has a vanishing moment map. We are thus imposing $g - 2$ D-term constraints, where g is the number of gauge groups. As in 3+1 dimensions, we can impose simultaneously the D-term constraints and the corresponding U(1) gauge transformations by modding out by the complexified gauge group. In other words, we are modding by the $g - 2$ dimensional subspace of the natural $(\mathbb{C}^*)^g$ action that is in the kernel of the matrix

$$C = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ k_1 & k_2 & \cdots & k_{g-1} & k_g \end{pmatrix}. \tag{2.3}$$

We do not need to divide by the remaining U(1) gauge field, since the last equation just determines the value of the auxiliary field σ . However, through its CS coupling with the overall gauge field, it leaves a discrete symmetry \mathbb{Z}_k , where $k = \text{gcd}(\{k_a\})$.

It is then easy to show that the resulting moduli space is a toric Calabi-Yau four-fold. The solution of the F-term constraints is a $g + 2$ dimensional toric Calabi-Yau variety called **the master space** [32, 33]. All the complexified gauge groups, except the overall one, act on this variety as non-trivial \mathbb{C}^* toric actions. By modding out by the $(\mathbb{C}^*)^{g-2}$ kernel of C we obtain, as promised, a Calabi-Yau four-fold.⁴ The moduli space is obtained by a further modding by the remaining discrete \mathbb{Z}_k symmetry. This moduli space is interpreted as the transverse space to one M2-brane in M-theory which probes the CY four-fold.

⁴The resulting variety is Calabi-Yau because the vectors of charges in \mathbb{C}^{g-2} are traceless by construction.

As shown in [30], the non-abelian mesonic moduli space is generically the N -fold symmetrized product of the Calabi-Yau four-fold.

In this paper, following the notation introduced in [30], we use the Calabi-Yau three-fold Y appearing in the 3+1 dimensional theory to identify the 2+1 dimensional theory. A Chern-Simons theory associated with the tiling for the Calabi-Yau three-fold Y with Chern-Simons parameters k_a is denoted $\tilde{Y}_{\{k_a\}}$.

3 The Calabi-Yau four-fold

The natural and immediate question is how to determine the toric data of the Calabi-Yau singularity. We emphasize that, in principle, all the information about the variety is encoded in the computation of the moduli space as a solution of F- and D-term constraints given in the previous section as a symplectic quotient of the master space. The toric properties of the master space have been studied in [32, 33, 41] and can be used to extract the toric data of the four-fold. In particular we always have a forward algorithm based on finding the kernel of the matrix of charges in the symplectic quotient description of the moduli space. The master space is however quite big and the previous analysis is done on a case by case basis. We therefore need to look for a more efficient algorithm to compute the toric diagram of the four-fold from the tiling data. Recall that, in 3+1 dimensions, the analogous algorithm is provided by the Kasteleyn matrix [24, 25]. We next show that a simple modification of the prescription works also in 2+1 dimensions.

We first explain the prescription using the simple ABJM theory as an example.

3.1 Toric data of the ABJM theory

The tiling and quiver of this theory are depicted in Figure 1, and the superpotential is given in (2.1). We set the Chern-Simons levels to $(k, -k)$. In the abelian theory the F-terms are trivial and there are no D-terms to be imposed. The remaining discrete symmetry is given by a \mathbb{Z}_k subgroup of the gauge group. The moduli space is then $\mathbb{C}^4/\mathbb{Z}_k$.

The fundamental cell of the periodic graph is shown in Figure 2. The toric diagram of the moduli space is computed by taking the permanent⁵ of a certain adjacency matrix of the graph. The rows of this matrix correspond to black nodes and the columns correspond to white nodes. An element of the matrix therefore selects a black and a white vertex in the graph. If there is no connection between them, then the element is zero. If there is an edge between them, then the matrix element is one. If the edge crosses the boundaries of the fundamental domain, then one needs to multiply by a corresponding weight x or y (or x^{-1} or y^{-1} , depending on the orientation of the edge). The resulting matrix is called the Kasteleyn matrix.

The permanent of the matrix is a Laurent polynomial of two variables x and y . For 3+1 dimensional theories, the Newton polygon of this polynomial gives the toric diagram of the Calabi-Yau threefold. This polygon is constructed by taking the convex hull of a

⁵The permanent is similar to the determinant: the signatures of the permutations are not taken into account and all terms come with a + sign. One can also use the determinant but then certain signs must be introduced [24, 25].

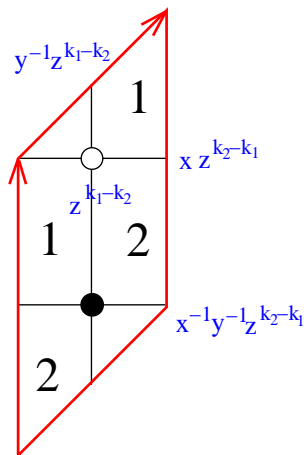


Figure 2. Fundamental cell of the \mathbb{C}^4 brane tiling. The weights of the four edges are shown in blue.

set of points in an integer lattice. These points are in one-to-one correspondence with the terms in the polynomial and their position in the lattice is given by the exponents of the x and y weights. The resulting two-dimensional diagram is the toric diagram of the threefold moduli space (recall that the triviality of the canonical class restricts the endpoints of the vectors to be on a plane).

In order to avoid dividing by the overall $U(1)$, we modify this simple algorithm by introducing a third weight z . This makes it possible to obtain three dimensional toric diagrams which in turn define Calabi-Yau fourfolds. We give a bifundamental field between gauge groups a and b a weight $z^{k_a-k_b}$. Here k_a and k_b are the Chern-Simons levels for the groups. The full set of weights for the ABJM theory is shown in Figure 2.

Since the fundamental cell of the ABJM theory contains only one black and one white node, the Kasteleyn matrix is 1×1 ,

$$K = \frac{\tilde{z}}{y} + \tilde{z} + \frac{x}{\tilde{z}} + \frac{1}{xy\tilde{z}} \tag{3.1}$$

where $\tilde{z} = z^{k_1-k_2} = z^{2k}$. By setting $\tilde{z} = 1$, the 2d Newton polygon gives the conifold toric diagram which is a square up to an $SL(2, \mathbb{Z})$ transformation. If we keep \tilde{z} , then we obtain a 3d diagram which is shown in Figure 3. This is a tetrahedron and it is the toric diagram for \mathbb{C}^4 when $k = 1$. As the level k increases, the tetrahedron gets stretched in the third dimension. In toric geometry, such rescaling is equivalent to orbifolding and thus the Kasteleyn algorithm reproduces the $\mathbb{C}^4/\mathbb{Z}_k$ ABJM moduli space.

The same algorithm works for more complicated tilings as is demonstrated in many explicit examples in this paper. In the general case, however, an issue arises: after proper normalization, the resulting toric diagrams are typically not at integer points in the 3d lattice. This makes it sometimes harder to identify the moduli spaces. This problem may be addressed by an appropriate “gauge transformation” of the weights on the tiling edges such that they give rise to integer toric diagrams.

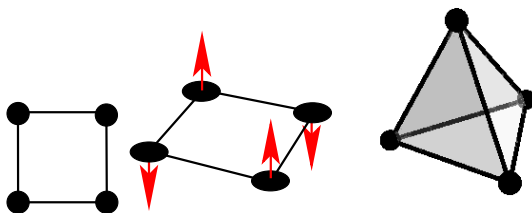


Figure 3. (i) Toric diagram for the conifold. (ii) Introducing the level translates the points as shown by the arrows. (iii) The resulting toric diagram is that of \mathbb{C}^4 .

3.2 Kasteleyn matrix and perfect matchings

Before proceeding, let us discuss the useful concept of perfect matching. We briefly review it because it is used in the following. The reader is referred to [24, 25, 42] for a comprehensive introduction. A perfect matching is a subset of edges in the tiling, or equivalently a subset of elementary fields, that covers each vertex exactly once. The permanent of the Kasteleyn matrix is precisely a generating function for perfect matchings. Each perfect matching is therefore associated with an integer point in the toric diagram. We can see this by writing a refined Kasteleyn matrix by inserting also the field corresponding to each edge. For the ABJM theory we have for example

$$K = B_1 \frac{\tilde{z}}{y} + B_2 \tilde{z} + A_1 \frac{x}{\tilde{z}} + A_2 \frac{1}{xy\tilde{z}} \quad (3.2)$$

Each term corresponds to one of the four perfect matchings, $p_1 = A_1, p_2 = A_2, p_3 = B_1, p_4 = B_2$. The ABJM theory is particularly simple since the Kasteleyn matrix is one by one. In a more general situation, each monomial in $\text{perm} K$ would be the product of the fields belonging to a given perfect matching: $\prod_{X_i \in p_\alpha} X_i$. We discuss many explicit examples below.

The importance of perfect matchings comes from the fact that they generate the master space [32]. In fact, as shown in [24, 25], the perfect matchings p_α parameterize the solutions of the F-term conditions through the formula

$$X_i = \prod_{\alpha=1}^c p_\alpha^{P_{i\alpha}}. \quad (3.3)$$

Here the matrix P contains entries which are either 0 or 1, encoding whether a field X_i in the quiver is in the perfect matching p_α :

$$P_{i\alpha} = \begin{cases} 1 & \text{if } X_i \in p_\alpha, \\ 0 & \text{if } X_i \notin p_\alpha. \end{cases} \quad (3.4)$$

For the conifold theory the situation is trivial, since there is a one to one correspondence between fields and perfect matchings. The master space is just \mathbb{C}^4 . In general, there are more perfect matchings than fields, and the parametrization (3.3) is defined only modulo some redefinition of the p_α . We get a description of the master space as a Gauged Linear Sigma Model (GLSM), or symplectic quotient: $\mathbb{C}^c / (\mathbb{C}^*)^{c-2-g}$. The vectors of charges

appearing in this description can be read off from the tiling as the linear relations satisfied by the perfect matchings $\sum_{\alpha} Q_{\alpha} p_{\alpha} = 0$ considered as formal linear combinations of edges.

We emphasize the conceptual importance of the perfect matchings. In 3+1 dimensions, they correspond to the integral points in the toric diagram including the internal ones and including multiplicities (internal points have multiplicity greater than 1 and external points have multiplicity 1. Points on the boundary of the toric diagram are binomial coefficients [43]). The perfect matchings corresponding to the external points of the 2d toric diagram, with multiplicity one, can be used to parametrize all the non-anomalous charges of the 3+1 dimensional theory [37, 44, 45]. In 2+1 dimensions, the 3d toric diagram is a split version of the 2d one, where some multiplicities have been lifted, keeping the total number of perfect matchings fixed. The perfect matchings still correspond to integer points in the 3d toric diagram. As we show, at least in all the examples considered in this paper, the external perfect matchings of the 3d toric diagram can be used to parametrize the $g + 2$ charges of the 2+1 dimensional theory. Finally, if we grow in dimensions with the purpose of studying the master space, we discover that all multiplicities have been lifted and the perfect matchings correspond exactly to the external point of the toric diagram for the coherent component of the master space that is a $g - 2$ dimensional Calabi-Yau singularity [32].

It is also important to note that the total number of perfect matchings is preserved by our construction. An important corollary is that only 3d toric diagrams which have a 2d projection such that the resulting 2d toric diagram corresponds to a consistent 3+1 dimensional theory⁶ are represented by a 2d tiling.

4 The Hilbert series

In this paper we describe the properties of the Calabi-Yau four-fold by means of its Hilbert series. Recall that the Hilbert Series has many equivalent interpretations:

- Mathematically, it is the generating function for holomorphic functions on the Calabi-Yau $X = C(H)$.
- Physically, from the field theory point of view, it is the partition function that counts chiral mesonic operators for the theory on one membrane. Thanks to the beautiful structure of the moduli space, which is a symmetric product, it is also the partition function for single trace chiral mesonic operators for large N [47].
- Finally, from the point of view of the compactification on $AdS_4 \times H$, it is the generating function for KK chiral multiplets [48]. In fact every holomorphic function on X descends to to an eigenvector of the Laplacian on the base H .

When the four-fold X is toric we can refine the Hilbert series with four weights t_i corresponding to 4 global $U(1)$ symmetries, some of which may be subgroups of a bigger

⁶See [46] for the most updated discussion on consistency of 2d tilings as giving rise to 3+1 dimensional theories.

non-abelian symmetry group. The meaning of the refined Hilbert series is to count operators according to their global and R-charges. One important geometrical property of the Hilbert Series is that, for $t_i \rightarrow 1$, it computes the volume of the base H [35]. In fact, setting $t_i = e^{-\mu b_i}$ we have for $\mu \rightarrow 1$

$$g(t_i; X) \sim \frac{\text{Vol}(b_i)}{\mu^4} + \dots \tag{4.1}$$

where the numerator is the volume of the family of Sasaki manifold with Reeb vector $R = \{b_1, b_2, b_3, b_4\}$.

At the superconformal fixed point there is a notion of the exact R-symmetry, which sits in the superconformal algebra. The exact R-symmetry corresponds to the Reeb vector that defines a Calabi-Yau metric on the cone. As shown by [36], the Reeb vector can be found by minimizing the function $\text{Vol}(b_i)$. The minimization is done on the three parameter set of b_i that give R-charge 2 to the holomorphic top form of the CY. The minimization, in turn, determines the exact R-symmetry. From the Hilbert series we can then compute the spectrum of dimensions of all mesonic operators, which agrees with the KK computation. For a $U(N)$ theory this is the complete spectrum of chiral operators. The R-charges of mesonic operators can be read from minimization.

Let us see the Hilbert series and minimization in action for the simple ABJM theory. There are 4 perfect matchings in correspondence with 4 quiver fields. One can assign to each point in the toric diagram a perfect matching, or alternatively, to each point in the toric diagram a quiver field. The moduli space is \mathbb{C}^4 , parametrized by four coordinates, A_1, A_2, B_1, B_2 . The global symmetry is $U(4)$ with rank 4 and one can assign four fugacities $t_i, i = 1 \dots 4$, each counting the number of fields of type i . The refined Hilbert series is particularly simple since the moduli space is freely generated by four variables

$$g\left(t_i, \tilde{\mathcal{C}}_{\{1, -1\}}\right) = \frac{1}{(1-t_1)(1-t_2)(1-t_3)(1-t_4)} \tag{4.2}$$

To compute R-charges we introduce chemical potentials $t_i = \exp(-\mu b_i)$ and take the limit $\mu \rightarrow 0$, with the coefficient of the most singular piece,

$$\lim_{\mu \rightarrow 0} \mu^4 g\left(e^{-\mu b_i}, \tilde{\mathcal{C}}_{\{1, -1\}}\right) = \frac{1}{b_1 b_2 b_3 b_4}, \tag{4.3}$$

and impose the CY condition $b_1 + b_2 + b_3 + b_4 = 2$. This condition comes from the fact that we require the top holomorphic form $dt_1 \wedge dt_2 \wedge dt_3 \wedge dt_4$ to have R-charge 2. Minimizing this expression we find $b_i = 1/2, i = 1 \dots 4$, leading to R-charges 1/2 per each field which is the canonical dimension for a scalar field in 2+1 dimensions. Since the global symmetry has a non-abelian factor we expect no contributions from the corresponding chemical potentials. We set $t_1 = tx_1, t_2 = tx_2/x_1, t_3 = tx_3/x_2, t_4 = t/x_3$, where t corresponds to the $U(1)_R$ symmetry, and x_1, x_2, x_3 are weights of the non-abelian $SU(4)$ symmetry. We further set $t = e^{-\mu b}$ and the CY condition takes the form $4b = 2$, which immediately gives the desired answer $b = 1/2$.

In more complicated examples, the Calabi-Yau four-fold will be given by a symplectic quotient $\mathbb{C}^d / (\mathbb{C}^*)^{d-4}$ in some ambient space of dimension d . We can then use an integral

Molien formula to compute the Hilbert series

$$\oint \prod_{i=1}^{d-4} \frac{dz_i}{2\pi i z_i} \frac{1}{\prod_{a=1}^d (1 - \tilde{t}_a Z_a)} \tag{4.4}$$

where $\tilde{t}_a = \tilde{t}_a(t_i)$ is a convenient parametrization of the coordinates in \mathbb{C}^d in terms of the four toric charges and $Z_a = Z_a(z_i)$ denotes the monomial weight of the a -th coordinate in terms of the $(\mathbb{C}^*)^{d-4}$ group. We refer to appendix A for a detailed explanation of the Molien formula.

We always have such a description of the Calabi-Yau four-fold as a symplectic quotient. One description is familiar from toric geometry [49] and it is obtained from the toric data provided by the Kasteleyn matrix. Another description follows from the explicit construction of the moduli space in field theory: the CY four-fold can be written as a symplectic quotient of the coherent component of the master space of dimension $g + 2$, which is itself a symplectic quotient of the space of perfect matchings [32, 33]. Since the number of perfect matching is large this description is sometimes cumbersome, even if more physical than the purely geometric one. In some lucky cases, we will be able to write the master space or the moduli space as a set of algebraic equations describing a complete intersection variety; in all these cases the computation of the Hilbert series simplifies.

4.1 Baryonic charges

Now recall that the AdS/CFT correspondence applies to $SU(N)$ theories. This means that the number of global symmetries of our theories is larger than four; it is actually $g + 2$, where g is the number of gauge groups.⁷ For the ABJM theory $g + 2 = 4$ is the number of toric symmetry of the four-fold and there is no baryonic charge, but, in a general tiling, $g + 2 > 4$ and there is plenty of baryonic charges. The mesonic spectrum does not depend on the new charges, but the baryonic spectrum does. Using only the Hilbert series and minimization we cannot determine the R-charges of all fields.

This is usually solved by looking explicitly at baryons. Recall that in the AdS/CFT correspondence baryons appear as wrapped branes. In 2+1 dimensions, a simple scaling argument says that, in order to have objects with dimensions proportional to N , we need to wrap M5 branes on five-cycles in H . The R-charge of a five-brane wrapped on the cycle Σ_5 in H is given as a normalized volume by the familiar formula [50]

$$R \equiv \Delta = \frac{\pi N \text{Vol}(\Sigma_5)}{6 \text{Vol}(H)}. \tag{4.5}$$

We refer to the particular ratio of volumes appearing in the previous formula as *normalized volume* in the rest of the paper. As usual in toric geometry, the five-cycles Σ_5 are associated with divisors D_a in X and with the external points v_a in the 3d toric diagram [49]. The

⁷ We see that the number of global symmetries that can mix with the R-charge is greater than the analogous number in the 3+1 dimensional theory associated with the same tiling, which is the number of external points of the 2d toric diagram minus three. The reason is that in 3+1 dimensions some global symmetries are anomalous while in 2+1 dimensions there is no anomaly.

normalized volumes can be easily computed using geometrical methods for all toric Calabi-Yau four-folds X . The technical results on computation of volumes are given in appendix A.

As is well known, an analogous computation in 3+1 dimensions gives the exact R-charges of all elementary fields. The reason is that all wrapped branes appear as baryons made with elementary fields. There is indeed a set of fields in the 3+1 dimensional quiver that is directly associated with the divisors D_a and whose R-charge can be computed by using normalized volumes. All the other elementary fields are associated with integer linear combinations of divisors and their R-charge is known by the additive property of the charge [37, 44, 45].

We analyze in detail many examples in 2+1 dimensions. We note that in all our examples the elementary fields can be set to be in correspondence with linear combinations of divisors, or equivalently linear combinations of five-cycles, although not all such combinations appear. The value of the exact R-charges corresponding to a five-brane wrapped on a divisor base is given by formula (4.5). Unfortunately, there is no efficient tool in 2+1 dimensions to compute the exact R-charge at the fixed point, so the comparison with field theory can be only qualitative. We also note that not all the wrapped branes appear as elementary fields; the existence of other states in the CS theory remains to be properly understood.

We conclude this section with a technical remark. In all our examples the external perfect matchings in the 3d toric diagram can be used to parametrize the $g + 2$ global charges, exactly as in 3+1 dimensions the external perfect matchings of the 2d toric diagram parametrize the non anomalous global charges. For the generic case of a regular 2d toric diagram which becomes a regular 3d toric diagram, we expect exactly $g + 2$ external points. Recall indeed that the number of gauge groups is given by the area of the 2d toric diagram that, by Pick's theorem [49], is $g = n + 2i - 2$ where n is the perimeter of the toric diagram (the number of integer points on the boundary) and i is the number of integer internal points. In a regular 2d diagram without integer points on the sides, the external points have multiplicity one and they remains external points in the 3d diagram. An internal point has multiplicity, but in the 3d diagram only two of the split points appear as external. The counting now generically reproduce $g + 2 = n + 2i$. The external points, and the corresponding perfect matchings can be then used to parametrize the global charges. More care should be used in cases where there are points on the sides, and this will be analyzed case by case in the following examples.

5 Examples

5.1 The $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ theory

This example is special since it is the simplest model with a spectrum of non-trivial scaling dimensions. It has two gauge groups just like the modified conifold theory of section 3.1, and 6 fields — slightly more than the 4 of section 3.1 but more crucially it has a cubic superpotential. This is to be contrasted with the modified conifold theory which has a quartic superpotential.

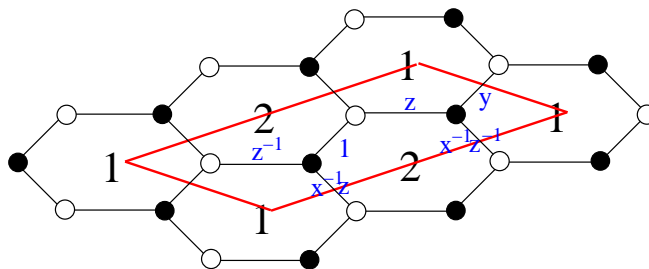


Figure 4. Brane tiling for $\mathcal{C} \times \mathbb{C}$ with edge weights around the two black nodes.

Let us re-consider the modified conifold case briefly. Symmetry reasoning leads to the natural scaling dimension $1/2$ for all basic fields consistent with the value for a free field in 2+1 dimensions. This argument implies that the simplest interaction term of the superpotential in 2+1 dimensions is quartic, different from the simplest interaction term in 3+1 dimensions which is cubic. Indeed the free field scaling dimension in 3+1 dimensions is 1 and to get a superpotential with an interaction term of dimension 3 we would take a cubic term. By using this simple scaling argument we learn that any interaction term in 2+1 dimensions which is not quartic must lead to non-trivial scaling dimensions as it is not possible to assign scaling $1/2$ to all fields and still get a total scaling of 2. At least one of the fields must exhibit some strong coupling effect. The simplest interaction term with these property is therefore the cubic term. We therefore crown the current example as the simplest model with a spectrum of non-trivial scaling dimensions for chiral operators. Below we will list the scaling dimensions and indicate a method for computing them using simple tools in toric geometry and tools from the theory of two dimensional tilings.

Figure 4 shows the tiling for this theory. The theory has two gauge groups, two adjoint fields Φ_i and four chiral fields $A_i, B_i, i = 1, 2$ transforming in the (N, \bar{N}) and (\bar{N}, N) representation of the gauge group, respectively, and interacting with the superpotential

$$W = \Phi_1(A_1 B_2 - A_2 B_1) + \Phi_2(B_2 A_1 - B_1 A_2) \tag{5.1}$$

The moduli space of the 2+1 dimensional theory is the conifold times the complex plane [30]. Since the number of gauge group is 2 there is no D-term to be imposed and the moduli space of the CS theory coincides with the coherent component of the master space, which is indeed four-dimensional. We note that the master space is reducible. On the coherent component of the moduli space the relation $\phi_1 = \phi_2$ is valid [32, 51, 52] and we will generically denote the independent adjoint field as ϕ . A similar remark applies to other orbifold examples in this paper.

To compute the toric diagram for CS levels $\{1, -1\}$ we write the Kasteleyn matrix,

$$K = \begin{pmatrix} z^{-1} + zx^{-1} & 1 \\ y & z + z^{-1}x^{-1} \end{pmatrix} \tag{5.2}$$

and the permanent gives

$$\text{perm } K = 1 + \frac{1}{z^2 x} + \frac{z^2}{x} + \frac{1}{x^2} + y. \tag{5.3}$$

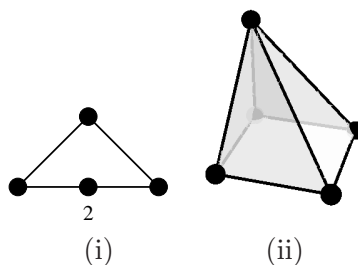


Figure 5. (i) The 2d toric diagram for $\mathbb{C}_2/\mathbb{Z}_2 \times \mathbb{C}$, denoted below by \mathcal{T}_2 . (ii) The 3d toric diagram for $\mathbb{C} \times \mathbb{C}$, denoted below by \mathcal{T}_3 . The internal point of multiplicity 2 in \mathcal{T}_2 splits into two external points in \mathcal{T}_3 .

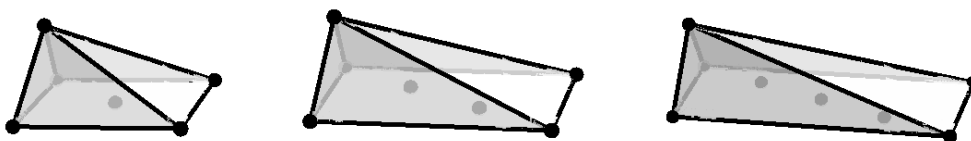


Figure 6. The 3d toric diagram for $\mathbb{C} \times \mathbb{C}$ orbifolds, with \mathbb{Z}_2 , \mathbb{Z}_3 and \mathbb{Z}_4 , respectively

This result shows that the double point of the original toric diagram of $\mathbb{C}_2/\mathbb{Z}_2$ gets separated in the third dimension, as depicted in Figure 5. However, in order to get the minimal volume we need a redefinition $z \rightarrow z^{1/4}$. Without this we would get a toric diagram that has internal points and corresponds to an orbifold of the $\{1, -1\}$ theory. Examples of the toric diagram for CS levels $\{2, -2\}$, $\{3, -3\}$, $\{4, -4\}$, respectively, are shown in Figure 6.

Next, we look at quiver fields and perfect matchings in order to prepare the ground for the scaling dimensions. For this example there are 6 quiver fields $\phi_1, A_1, B_1, \phi_2, A_2, B_2$. We first use a symmetry argument to compute the scaling dimensions, before making a detailed analysis. The \mathbb{C} part of the moduli space is not changing in this problem and therefore we expect the fields parametrizing it, ϕ , to have the trivial scaling dimension $1/2$. The rest of the fields, A, B are expected to have the same scaling dimension due to the $SU(2)$ symmetry that acts on the index i of both A_i and B_i , as well as the \mathbb{Z}_2 symmetry of the quiver theory that exchanges A and B together with charge conjugation. Since the superpotential is of the form ϕAB and has a scaling 2, we conclude that A and B have a scaling dimension $3/4$.

Let us confirm this with a more detailed analysis. The permanent of the Kasteleyn matrix shows that there are 5 perfect matchings $p_\alpha, \alpha = 1 \dots 5$. From the point of view of the 2d toric diagram, p_1, p_2, p_3 correspond to the external points on \mathcal{T}_2 in Figure 5 while p_4, p_5 corresponds to the single point on the boundary of \mathcal{T}_2 of Figure 5, in between p_1 and p_2 . In the 3d toric diagram \mathcal{T}_3 , p_4, p_5 split.

On the coherent component of the moduli space we find the parameterization of fields in terms of perfect matchings [25]

$$A_1 = p_1 p_4, \quad A_2 = p_2 p_4, \quad B_1 = p_1 p_5, \quad B_2 = p_2 p_5, \quad \phi_1 = \phi_2 = p_3. \quad (5.4)$$

The previous relation is easily obtained from the permanent of the Kasteleyn matrix with

the insertion of elementary fields

$$\begin{aligned} \text{perm} \left(\begin{array}{cc} A_1 z^{-1} + B_2 z x^{-1} & \phi_2 \\ \phi_1 y & B_1 z + A_2 z^{-1} x^{-1} \end{array} \right) \\ = A_1 B_1 + A_1 A_2 \frac{1}{z^2 x} + B_1 B_2 \frac{z^2}{x} + B_2 A_2 \frac{1}{x^2} + \phi_1 \phi_2 y. \end{aligned} \quad (5.5)$$

by recalling that each monomial corresponds to a perfect matching. For example A_1 belongs to the perfect matchings p_1 and p_4 and this leads to the relation $A_1 = p_1 p_4$ and similarly for the other fields.

From this we learn that the external points of \mathcal{T}_2 , denoted p_1, p_2, p_3 , carry the information on the complex directions of $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$, while the internal points of \mathcal{T}_2 , denoted p_4, p_5 are carrying the information on the A quantum number and B quantum number, respectively, namely, p_4 counts the number of A 's and p_5 counts the number of B 's. All together there are 4 conserved charges corresponding to each point in the toric diagram \mathcal{T}_2 . The two perfect matchings p_4, p_5 are associated with the internal point and therefore do not carry a quantum number which is independent. Instead we can introduce a conserved quantum number which in 3+1 dimensions is called the baryonic charge which counts the number of A 's minus the number of B 's. In 2+1 dimensions all 4 charges are mesonic. When the number of gauge groups is more than 2 we can find additional baryonic charges and they play a role in the next set of examples.

The moduli space is given by the collection of these 5 perfect matchings subject to the relation $p_1 + p_2 = p_4 + p_5$ which give rise to the charge vector $(1, 1, 0, -1, -1)$. We see that the quiver fields ϕ 's, A 's and B 's are gauge invariants with respect to this charge. Furthermore, (5.4) gives the conifold relation

$$A_1 B_2 = A_2 B_1. \quad (5.6)$$

This relation is the F-term relation corresponding to the adjoint fields ϕ . To compute the fully refined Hilbert series it is enough for this case to use the property that the moduli space $\mathcal{C} \times \mathbb{C}$ [30] is a complete intersection. We further exploit the global symmetry of this model, which coincides with the symmetry of the moduli space, $SU(2)_1 \times SU(2)_2 \times U(1)_1 \times U(1)_2$. Let us introduce fugacities $t_{1,2}$ for $U(1)_{1,2}$, and $x_{1,2}$ for $SU(2)_{1,2}$, respectively. With these fugacities and with the help of the charge vector we assign fugacities $t_1 x_1, t_1/x_1, t_2, t_1 x_2, t_1/x_2$ to the perfect matchings, p_1, p_2, p_3, p_4, p_5 , respectively. From this and from (5.4) it follows that the quiver fields $\phi_1, A_1, B_1, \phi_2, A_2, B_2$ have fugacities $t_2, t_1^2 x_1 x_2, t_1^2 x_2/x_1, t_2, t_1^2 x_1/x_2, t_1^2/x_1 x_2$, respectively. This assignment puts $p_{1,2}$ and $p_{4,5}$ in the $[1; 0]$ and $[0, 1]$ representations of $SU(2)_1 \times SU(2)_2$, respectively, while the A 's and B 's transform in the $[1; 1]$ representation. With these preparations we are now ready to write down the refined Hilbert series and it takes the form

$$\begin{aligned} g \left(t_1, t_2, x_1, x_2; \widetilde{\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}_{\{1, -1\}}} \right) \\ = \frac{1 - t_1^4}{(1 - t_1^2 x_1 x_2)(1 - t_1^2 x_2/x_1)(1 - t_2)(1 - t_1^2 x_1/x_2)(1 - t_1^2/x_1 x_2)}, \end{aligned} \quad (5.7)$$

where the denominator takes into account the generators, A 's, B 's and ϕ , and the numerator takes into account the relation (5.6). Alternatively, for generalization to cases in which the moduli space is not a complete intersection we can use a Molien integral,

$$g\left(t_1, t_2, x_1, x_2; \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}_{\{1,-1\}}\right) = \oint_{|z|<1} \frac{dz}{2\pi iz} \frac{1}{(1-t_1x_1z)(1-t_1z/x_1)(1-t_2)(1-t_1x_2/z)(1-t_1/x_2z)}. \tag{5.8}$$

In this formula the denominator takes into account the six perfect matchings and the integration takes into account the existence of the linear relation $p_1 + p_2 = p_4 + p_5$.

Next we are ready to compute the volume of the SE_7 . Set $t_1 = e^{-\mu b_1}, t_2 = e^{-\mu b_2}$. Due to the non-abelian symmetry $SU(2)_1 \times SU(2)_2$ we do not expect the corresponding fugacities to affect the formula for the volume and can therefore safely set their values to 1 at the extremal point. Alternatively, if this argument is not trusted, we can assume a dependence on these two variables, get a more complicated formula, and find that the extremum at $x_1 = x_2 = 1$ follows. Taking the limit we find

$$\lim_{\mu \rightarrow 0} \mu^4 g\left(e^{-\mu b_1}, e^{-\mu b_2}, 1, 1; \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}_{\{1,-1\}}\right) = \frac{1}{4b_1^3 b_2} \tag{5.9}$$

The CY condition now sets the sum over all external perfect matchings in the 3d toric diagram \mathcal{T}_3 to be $4b_1 + b_2 = 2$, from which the minimization gives $b_1 = 3/8, b_2 = 1/2$. The four points p_1, p_2, p_4, p_5 have charge $3/8$, equal as expected by symmetry of the toric diagram, and p_3 has charge $1/2$. Tracing this back to the quiver fields we find a reassuring result that the R charge for the adjoint fields $\phi_{1,2}$ is $1/2$ consistent with the canonical dimension for a scalar field in 2+1 dimensions. The fields A and B have charge $3/4$ which is consistent with the cubic superpotential.

Since there are no baryonic charges, the R-charges of the fields are uniquely determined by the minimization on the mesonic Hilbert series. For consistency we check that the values of charges for p_i are given by the normalized volumes of the corresponding divisors D_i in the geometry. By symmetry, D_1, D_2, D_4, D_5 will have the same volume. The normalized volumes can be computed as explained in the appendix. The explicit computation is actually superfluous because we expect result $1/2$ for the free factor \mathbb{C} corresponding to D_3 ; the fact that normalized volumes add up to 2 then fix the value of the others to $3/8$. Let us do nevertheless the explicit computation to prepare for more complicated examples. Consider the Molien integral (5.8). The five terms in the denominator are the weights corresponding to the five perfect matchings p_i and therefore to the five divisors D_i . Formula (A.3) instructs us to compute the Hilbert series corresponding to D_i by computing the same Molien integral with the insertion of (the inverse of) the weight corresponding to the divisor D_i . We have (setting by symmetry $x_1 = x_2 = 1$),

$$g\left(D_{1,2}; t_1, t_2; \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}_{\{1,-1\}}\right) = \oint_{|z|<1} \frac{dz}{2\pi iz} \frac{(t_1 z)^{-1}}{(1-t_1 z)^2 (1-t_2)(1-t_1/z)^2},$$

$$g\left(D_{4,5}; t_1, t_2; \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}_{\{1,-1\}}\right) = \oint_{|z|<1} \frac{dz}{2\pi iz} \frac{(t_1/z)^{-1}}{(1-t_1 z)^2 (1-t_2)(1-t_1/z)^2},$$

	SU(2) ₁	SU(2) ₂	U(1) _q	U(1) _R	fugacities
p_1	1	0	1/2	3/8	$t_1 x_1$
p_2	-1	0	1/2	3/8	t_1/x_1
p_3, ϕ_1, ϕ_2	0	0	-2	1/2	t^4/q^4
p_4	0	1	1/2	3/8	$t_1 x_2$
p_5	0	-1	1/2	3/8	t_1/x_2
A_1	1	1	1	3/4	$t^3 q x_1 x_2$
B_1	1	-1	1	3/4	$t^3 q x_1/x_2$

Table 1. Global charges for the perfect matchings for the quiver gauge theory on the M2 brane probing the modified $\mathbb{C}^2/\widetilde{\mathbb{Z}}_2 \times \mathbb{C}$ singularity.

$$g\left(D_3; t_1, t_2; \mathbb{C}^2/\widetilde{\mathbb{Z}}_2 \times \mathbb{C}_{\{1,-1\}}\right) = \oint_{|z|<1} \frac{dz}{2\pi i z} \frac{t_2^{-1}}{(1-t_1 z)^2 (1-t_2)(1-t_1/z)^2}. \quad (5.10)$$

The normalized volumes are then

$$\begin{aligned} \frac{g\left(D_i; e^{-\mu b_1}, e^{-\mu b_2}, 1, 1; \mathbb{C}^2/\widetilde{\mathbb{Z}}_2 \times \mathbb{C}_{\{1,-1\}}\right)}{g\left(e^{-\mu b_1}, e^{-\mu b_2}, 1, 1; \mathbb{C}^2/\widetilde{\mathbb{Z}}_2 \times \mathbb{C}_{\{1,-1\}}\right)} &= 1 + \frac{3}{8}\mu + \dots \quad i = 1, 2, 4, 5, \\ \frac{g\left(D_3; e^{-\mu b_1}, e^{-\mu b_2}, 1, 1; \mathbb{C}^2/\widetilde{\mathbb{Z}}_2 \times \mathbb{C}_{\{1,-1\}}\right)}{g\left(e^{-\mu b_1}, e^{-\mu b_2}, 1, 1; \mathbb{C}^2/\widetilde{\mathbb{Z}}_2 \times \mathbb{C}_{\{1,-1\}}\right)} &= 1 + \frac{1}{2}\mu + \dots \end{aligned} \quad (5.11)$$

as expected. From (5.4) we recover the previous assignment of R-charges for the elementary fields. Notice that no elementary field carries the charge 3/8 corresponding to the volumes of the four divisors D_1, D_2, D_4, D_5 . ϕ is associated with D_3 . The elementary fields A and B seem to be associated, consistently with the tiling prescription (5.4), to combinations of divisors $D_1 + D_4, D_2 + D_4, D_1 + D_5, D_2 + D_5$. It remains to explain whether there are BPS states in the theory associated to M5 branes wrapped on single divisors.

In view of the numerical values for the R-charges we can now define two new fugacities: t which counts the total R-charge, and q which is a conserved current that satisfies the condition that the total charge over external points of the 3d toric diagram is 0. We find $t_1^2 = t^3 q, t_2 = t^4/q^2$, such that the new form of the partition function (5.7) is

$$\begin{aligned} g\left(t, q, x_1, x_2; \mathbb{C}^2/\widetilde{\mathbb{Z}}_2 \times \mathbb{C}_{\{1,-1\}}\right) \\ = \frac{1 - t^6 q^2}{(1 - t^3 q x_1 x_2)(1 - t^3 q x_2/x_1)(1 - t^4/q^2)(1 - t^3 q x_1/x_2)(1 - t^3 q/x_1 x_2)}. \end{aligned} \quad (5.12)$$

We summarize the collection of charges in table 5.1.

For higher values of the CS couplings, $k, -k$ the moduli space can be determined as follows. It is a \mathbb{Z}_k action on $\mathcal{C} \times \mathbb{C}$ and the \mathbb{Z}_k action is easiest to see on the quiver fields, ϕ, A, B , with charges 0, 1, -1, respectively. Using table 5.1 this identifies with the weight of SU(2)₂, implying that x_2 is acted by \mathbb{Z}_k and all other weights are free. The resulting

Hilbert series is

$$\begin{aligned}
 g\left(t, q, x_1, x_2; \widetilde{\mathbb{C}^2/\mathbb{Z}_2} \times \mathbb{C}_{\{k, -k\}}\right) &= \\
 &= \frac{1 - t^6 q^2}{k} \sum_{j=0}^{k-1} \frac{1}{(1 - \omega^j t^3 q x_1 x_2)(1 - \omega^j t^3 q x_2/x_1)} \times \\
 &\quad \times \frac{1}{(1 - t^4/q^2)(1 - \omega^{-j} t^3 q x_1/x_2)(1 - \omega^{-j} t^3 q/x_1 x_2)}.
 \end{aligned}
 \tag{5.13}$$

5.2 The \widetilde{L}^{aba} theories

We proceed with the analysis of more models of the type L^{aba} [44, 45, 53–56]. A special subset of these theories are those with $a = 0$ corresponding to orbifolds of \mathbb{C}^3 with higher amount of supersymmetry in 3+1 dimensions, namely with 8 supercharges. Note, however, that while this is true in 3+1 dimensions, the presence of the CS couplings breaks the supersymmetry down to 4 supercharges in 2+1 dimensions, as can readily be seen by the simplest family of this class corresponding to $b = 2$ of the previous subsection.

The method used in the analysis of the previous subsection is with the explicit form of perfect matchings and their relation to the basic fields of the quiver. This method does not seem to be easily generalized to higher orbifold cases since the number of perfect matching grows exponentially with the order of the singularity b and the treatment using these degrees of freedom becomes cumbersome. Luckily we have a simple property of these models which saves the day. The coherent component of the master space for all of this class of theories is a complete intersection which is generated by $2a + 2b$ variables that are subject to $a + b - 2$ constraints. We further have luck on our side and use the large hidden symmetry that the master space has. The hidden symmetry of the coherent component of the master space is $SU(a) \times SU(b)$ which together with the explicit symmetry directions that add $U(1)^4$, one of which is baryonic and the rest are mesonic, leads to a global symmetry of rank $a + b + 2$ as expected from the dimension of the coherent component of the moduli space. We introduce 4 chemical potentials t_1, t_2, t_3, t_4 for the $U(1)$'s and chemical potentials x_i, y_j for the hidden symmetries. In the case that $a = 0$ there are only 3 $U(1)$'s, corresponding to 3 mesonic charges and the hidden symmetry group is $SU(b)$. Together they form a global symmetry of rank $b + 2$ which is again the expected value. The $2a + 2b$ generators transform as $[1, 0, \dots, 0; 0, \dots, 0]$, $[0, \dots, 0, 1; 0, \dots, 0]$, $[0, \dots, 0; 1, 0, \dots, 0]$, $[0, \dots, 0; 0, \dots, 0, 1]$ of $SU(a) \times SU(b)$. Each of these representations carries one of the 4 $U(1)$ charges. The relations are singlets of the hidden symmetries and carry charges with respect to two of the $U(1)$'s. With this information it is now possible to write the refined Hilbert series for the coherent component of the master space.

$$\begin{aligned}
 g(t_i, x_j, y_k; L_{aba}) &= (1 - t_1 t_2)^{a-1} (1 - t_3 t_4)^{b-1} \times \\
 &\quad PE\left[t_1[1, 0, \dots, 0; 0, \dots, 0] + t_2[0, \dots, 0, 1; 0, \dots, 0] \right. \\
 &\quad \left. + t_3[0, \dots, 0; 1, 0, \dots, 0] + t_4[0, \dots, 0; 0, \dots, 0, 1]\right]
 \end{aligned}
 \tag{5.14}$$

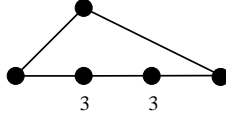


Figure 7. The 2d toric diagram for $\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$.

	$U(1)_1$	$U(1)_2$	$U(1)_3$	$U(1)_R$	fugacities	p.m.
A_1	1	-1	0	3/4	$t_1 x_1$	$p_1 q_1 \tilde{q}_2 \tilde{q}_3$
A_2	0	1	-1	3/4	$t_1 x_2 / x_1$	$p_1 \tilde{q}_1 q_2 \tilde{q}_3$
A_3	-1	0	1	3/4	t_1 / x_2	$p_1 \tilde{q}_1 \tilde{q}_2 q_3$
B_1	-1	1	0	3/4	t_2 / x_1	$p_2 \tilde{q}_1 q_2 q_3$
B_2	0	-1	1	3/4	$t_2 x_1 / x_2$	$p_2 q_1 \tilde{q}_2 q_3$
B_3	1	0	-1	3/4	$t_2 x_2$	$p_2 q_1 q_2 \tilde{q}_3$
ϕ_1, ϕ_2, ϕ_3	0	0	0	1/2	t_3	p_3

Table 2. Gauge charges for the generators of the theory on an M2 brane probing the modified $\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$ singularity.

For the case of $a = 0, b = n$ this equation is slightly modified and takes the form

$$g(t_i, x_j; \mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}) = \frac{(1 - t_1 t_2)^{n-1}}{(1 - t_3)} PE [t_1[1, 0, \dots, 0] + t_2[0, \dots, 0, 1]] \quad (5.15)$$

5.3 The $\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$ theories

Let us specialize to the case of $a = 0, b = 3$. The superpotential of the theory is

$$W = \phi_1(A_1 B_1 - B_3 A_3) + \phi_2(A_2 B_2 - B_1 A_1) + \phi_3(A_3 B_3 - B_2 A_2). \quad (5.16)$$

The Hilbert series for the coherent component of the master space is

$$g(t_1, t_2, t_3, x_1, x_2; \mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}) = \frac{(1 - t_1 t_2)^2}{(1 - t_3)} PE [t_1[1, 0] + t_2[0, 1]]. \quad (5.17)$$

Since this is a complete intersection moduli space it is possible to write the relations explicitly,

$$A_1 B_1 = A_2 B_2 = A_3 B_3. \quad (5.18)$$

Let us fix the CS coefficients to be $k_1, k_2, -k_1 - k_2$. Before treating the general case let us first take $k_1 = k, k_2 = 0$. To get the moduli space of the 2+1 dimensional theory we need to divide by the D term of the gauge group with CS equal to 0. We need to identify the charges of the 6 generators under this gauge group. This is done in table 2. A quick inspection shows that we need to integrate over the x_1 variable. Let us write the Hilbert series for the coherent component of the master space more explicitly, together with the integral

$$\begin{aligned} g(t_1, t_2, t_3, x_2; \mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}_{\{1,0,-1\}}) &= \\ &= \oint_{|x_1| < 1} \frac{dx_1}{x_1} \frac{(1 - t_1 t_2)^2}{(1 - t_3)(1 - t_1 x_1)(1 - t_1 x_2 / x_1)(1 - t_1 / x_2)(1 - t_2 / x_1)(1 - t_2 x_1 / x_2)(1 - t_2 x_2)} \end{aligned}$$

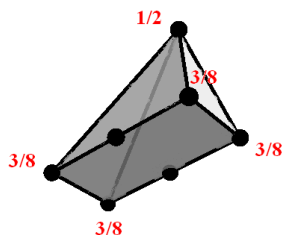


Figure 8. The 3d toric diagram for $L^{222} \times \mathbb{C}$. The R-charges are shown in red.

$$= \frac{1 - t_1^2 t_2^2}{(1 - t_1^2 x_2)(1 - t_1/x_2)(1 - t_2^2/x_2)(1 - t_2 x_2)(1 - t_3)}. \quad (5.19)$$

The mesonic branch of the 2+1 dimensional theory is then a complete intersection moduli space generated by 5 generators which are subject to one relation. Explicitly, we can find the generators to be $\phi_1 = \phi_2 = \phi_3$ parametrizing \mathbb{C} and $A_3, B_3, M_1 = A_1 A_2, M_2 = B_1 B_2$. Using (5.18) the relation can also be explicitly written,

$$M_1 M_2 = A_3^2 B_3^2. \quad (5.20)$$

This identifies the moduli space as the Z_2 orbifold of the conifold, otherwise known as L^{222} . To summarize, the mesonic moduli space of $\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}_{\{1,0,-1\}}$ is $L^{222} \times \mathbb{C}$.

We are now ready to compute the R-charges for the fields. On the mesonic branch of the 2+1 dimensional theory we just need to compute and minimize the volume of the seven manifold. This is done as usual by setting $t_1 = e^{-\mu b_1}, t_2 = e^{-\mu b_2}, t_3 = e^{-\mu(2-b_1-b_2)}, x_2 = e^{-\mu b_x}$ and computing the coefficient of the pole $1/\mu^4$. The restriction on the exponents comes from the fact that the holomorphic top form scales as $t_1 t_2 t_3$. We obtain the volume function

$$Z(b_1, b_2, b_x) = \frac{2(b_1 + b_2)}{(b_1 + b_2 - 2)(2b_2^2 + b_2 b_x - b_x^2)(-2b_1^2 + b_1 b_x + b_x^2)} \quad (5.21)$$

whose minimization gives $b_x = 0, b_1 = b_2 = 3/4$ corresponding to a dimension 3/4 for the fields A and B and 1/2 for ϕ . These are recorded in table 2. As in subsection 5.1 the charges can be computed by just using symmetry arguments and simple scalings. For reference, we notice that the toric diagram of $L^{222} \times \mathbb{C}$ has five external points corresponding to five volumes; as usual the factor \mathbb{C} has volume 1/2 and by symmetry and the fact that the normalized volumes add up to 2 we obtain the value 3/8 for the other four. With the parameterization given in table 2 the five points are labeled by $p_1, p_2, p_3, q_3, \tilde{q}_3$ and these values reproduce the R-charges of the elementary fields.

For generic CS parameters we need to divide by the gauge group $k_2 U(1)_1 - k_1 U(1)_2$. The refined Hilbert series takes the form

$$g\left(t_1, t_2, t_3, x; \mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}_{\{k_1, k_2, -k_1 - k_2\}}\right) = \quad (5.22)$$

$$= \oint_{|w|<1} \frac{dw}{w} \frac{(1 - t_1 t_2)^2}{(1 - t_3)(1 - t_1 w^{k_1 + k_2})(1 - t_1 x/w^{k_1})(1 - t_1/x w^{k_2})} \times$$

$$\times \frac{1}{(1 - t_2/w^{k_1 + k_2})(1 - t_2 w^{k_1}/x)(1 - t_2 w^{k_2} x)}$$

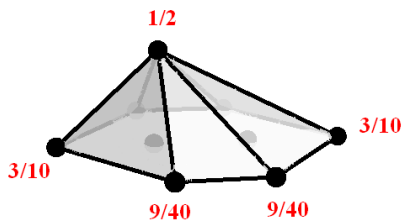


Figure 9. The 3d toric diagram for $\mathbb{C}^2/\widetilde{\mathbb{Z}}_3 \times \mathbb{C}_{\{1,1,-2\}}$. The R-charges are shown in red.

which corresponds to the change of variables $x_1 = w^{k_1+k_2}$, $x_2 = xw^{k_2}$ from table 2 and a subsequent integration over w which is the weight under the gauge group $k_2\text{U}(1)_1 - k_1\text{U}(1)_2$. We further need the fugacities for the perfect matchings. Using table 2 we can choose

$$p_1 : t_1, \quad p_2 : t_2, \quad p_3 : t_3, \quad q_1 : \frac{x_1}{x_2}, \quad q_2 : \frac{x_2}{x_1}, \quad q_3 : 1, \quad \tilde{q}_1 : \frac{1}{x_2}, \quad \tilde{q}_2 : 1, \quad \tilde{q}_3 : x_2 \quad (5.23)$$

An interesting case is $k_1 = k_2 = 1$ which has $\text{SU}(2)$ global symmetry. x is the corresponding weight. The 3d toric diagram has a hexagonal base and seven external points, which can set in correspondence with $p_1, p_2, p_3, q_1, q_3, \tilde{q}_1, \tilde{q}_3$. q_2 and \tilde{q}_2 are still internal point in the 3d toric diagram. A not difficult but long computation with the formulae in the appendix give the following values for the volumes: $1/2$ for p_1 , $3/10$ for p_2, p_3 and $9/40$ for $q_1, q_3, \tilde{q}_1, \tilde{q}_3$. As seen from the table this still correspond to R-charge $3/4$ for A, B, C and R-charge $1/2$ for ϕ .

5.3.1 A family of $\mathbb{C}^2/\widetilde{\mathbb{Z}}_{n+1} \times \mathbb{C}$ theories

An interesting family of theories can be taken to be $\mathbb{C}^2/\widetilde{\mathbb{Z}}_{n+1} \times \mathbb{C}_{\{1,0,\dots,0,-1\}}$. The first member of this class is studied in section 5.1 corresponding to $\mathbb{C}^2/\widetilde{\mathbb{Z}}_2 \times \mathbb{C}_{\{1,-1\}}$ with a $2+1$ dimensional moduli space $L^{111} \times \mathbb{C}$, where we have written the conifold $\mathcal{C} = L^{111}$ in a form which is more suitable for the generalization to higher n . The second member of this class is discussed in section 5.3 for $n = 2$. The general n case gives a moduli space which is $L^{nnn} \times \mathbb{C}$. This can be summarized with the following points. There are $3n + 3$ fields $\phi_i, A_i, B_i, \quad i = 1 \dots n + 1$. The superpotential is

$$W = \sum_{i=1}^{n+1} \phi_i (A_i B_i - B_{i-1} A_{i-1}), \quad (5.24)$$

where the index i is taken to be cyclic, modulo $n + 1$. The F-term equations on the coherent component of the master space take the form

$$A_1 B_1 = A_2 B_2 = \dots = A_n B_n = A_{n+1} B_{n+1}, \quad (5.25)$$

giving a complete intersection moduli space generated by $2n + 2$ variables subject to n constraints and another copy of \mathbb{C} , generated by $\phi_1 = \dots = \phi_{n+1}$. A choice of CS coefficients, $\{1, 0, \dots, 0, -1\}$ instructs to mod out by the $n - 2$ gauge groups which have CS level equal to 0. The resulting Hilbert series for the CY four-fold is

$$g \left(t_1, t_2, t_3, x; \mathbb{C}^2/\widetilde{\mathbb{Z}}_{n+1} \times \mathbb{C}_{\{1,0,\dots,0,-1\}} \right) = \frac{1 - t_1^n t_2^n}{(1 - t_1^n x)(1 - t_1/x)(1 - t_2^n/x)(1 - t_2 x)(1 - t_3)}.$$

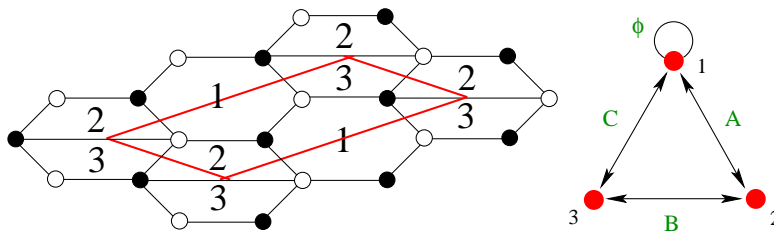


Figure 10. (i) Brane tiling for \widetilde{SPP} . The fundamental domain is shown in red. (ii) The corresponding quiver.

The gauge invariant generators of this moduli space are

$$M_1 = \prod_{i=1}^n A_i, \quad M_2 = \prod_{i=1}^n B_i, \quad A_{n+1}, \quad B_{n+1}, \quad \phi, \quad (5.26)$$

and the relation they satisfy is

$$M_1 M_2 = A_{n+1}^n B_{n+1}^n \quad (5.27)$$

corresponding to the announced result that the moduli space is $L^{nnn} \times \mathbb{C}$, where L^{nnn} is a non-chiral \mathbb{Z}_n orbifold of the conifold. The scaling dimensions can be determined by symmetry. There are 5 external points in the toric diagram of $L^{nnn} \times \mathbb{C}$, one corresponds to ϕ which has a scaling $1/2$. The other 4 are completely symmetric and get scaling dimension $3/8$ each. Finally, each of the fields A, B have a scaling dimension $3/4$.

5.4 The \widetilde{SPP} (L^{121}) revisited

The tiling and toric diagram are given in Figure 10. The 2d toric diagram is given in Figure 11. The theory has chiral fields ϕ, A_i, B_i, C_i indicated in Figure 10 and interacting with the superpotential

$$W = \phi(A_1 A_2 - C_2 C_1) - A_2 A_1 B_1 B_2 + C_1 C_2 B_2 B_1. \quad (5.28)$$

To compute the 3d toric diagram we write the Kasteleyn matrix (with all the fields inserted)

$$K = \begin{pmatrix} C_1 z^{k_1 - k_3} + C_2 z^{-k_1 + k_3} y & B_1 x^{-1} z^{-k_2 + k_3} + B_2 x^{-1} y z^{k_2 - k_3} \\ \phi & A_1 z^{-k_1 + k_2} + A_2 z^{k_1 - k_2} y \end{pmatrix}, \quad (5.29)$$

and the permanent gives

$$\begin{aligned} \text{perm } K = & A_1 C_1 z^{k_2 - k_3} + B_1 \phi \frac{z^{-k_2 + k_3}}{x} + C_1 A_2 z^{2k_1 - k_2 - k_3} y + A_1 C_2 z^{-2k_1 + k_2 + k_3} y \\ & + B_2 \phi \frac{z^{k_2 - k_3} y}{x} + A_2 C_2 z^{-k_2 + k_3} y^2. \end{aligned} \quad (5.30)$$

Let us denote the six perfect matchings corresponding to the six monomials in the previous expression as $p_1, p_3, q_2, q_1, p_4, p_2$ in the given order. From this expression we see

	p.m	fugacities
A_1	p_1q_1	$t_1^2x_1x_3$
A_2	p_2q_2	t_1^2/x_1x_3
C_1	p_1q_2	$t_1^2x_1/x_3$
C_2	p_2q_1	$t_1^2x_3/x_1$
B_1	p_3	t_2x_2
B_2	p_4	t_2/x_2
ϕ	p_3p_4	t_2^2

Table 3. Fields, perfect matchings and fugacities for the quiver gauge theory on the M2 brane probing the modified \widetilde{SPP} singularity.

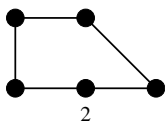


Figure 11. Toric diagram of the Suspended Pinch Point singularity ($xy = uv^2$).

that, for example, A_1 belongs to the perfect matchings p_1 and q_1 and therefore it can be parametrized as p_1q_1 . A similar computation for the other fields gives the result in table 3.

The coherent component of the master space is the space of the 6 perfect matchings which are subject to the single relation $p_1 + p_2 = q_1 + q_2$. This variety is \mathbb{C}^6 , parametrized by $p_1, p_2, p_3, p_4, q_1, q_2$ divided by the action $(1, 1, 0, 0, -1, -1)$ and it coincides with the conifold times \mathbb{C}^2 . This can be easily seen also from the F-term equations

$$A_1A_2 = C_1C_2, \quad \phi = B_1B_2 \tag{5.31}$$

The symmetry of this space is $SU(2)_1 \times SU(2)_2 \times SU(2)_3 \times U(1)_1 \times U(1)_2$, where all three $SU(2)$'s are hidden [32]. We introduce five independent weights t_i, x_i corresponding to the global symmetries of the master space as recorded in the table (t 's for $U(1)$'s and x 's for $SU(2)$'s). The perfect matchings $p_1, p_2, p_3, p_4, q_1, q_2$ have weights $t_1x_1, t_1/x_1, t_2x_2, t_2/x_2, t_1x_3, t_1/x_3$. The Hilbert series for the master space is then

$$\begin{aligned}
 g(t_i, x_i; SPP) &= \int \frac{dz}{2\pi iz} \frac{1}{(1-t_1zx_1)(1-t_1z/x_1)(1-t_2x_2)(1-t_2/x_2)(1-t_1x_3/z)(1-t_1/x_3z)} \\
 &= \frac{1-t_1^4}{(1-t_1^2x_1x_3)(1-t_1^2x_1/x_3)(1-t_1^2x_3/x_1)(1-t_1^2/x_1x_3)(1-t_2x_2)(1-t_2/x_2)}
 \end{aligned} \tag{5.32}$$

which is indeed the Hilbert series for the conifold times \mathbb{C}^2 .

The Newton polygon of $\text{perm } K|_{z=1}$ gives the Suspended Pinch Point toric diagram (Figure 11). By turning on the “magnetic flux”, the points of the diagram get pulled into the third dimension. Depending on the choices for k_i , we have several possibilities:

5.4.1 (i) $(k_1, k_2, k_3) = (1, -1, 0)$

gives the D_3 model whose 3d toric diagram is depicted in Figure 12(i): the two points q_i split and we obtain a toric diagram with six external points.

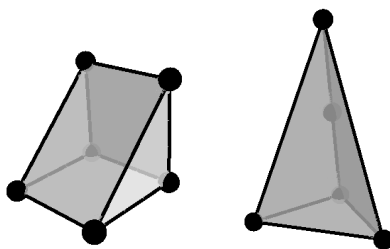


Figure 12. (i) Toric diagram for D_3 . (iii) Toric diagram for $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}^2$.

We need to mod out by the third gauge group. The Hilbert series for D_3 takes the form

$$g(t_1, t_2, x, y; \widetilde{SPP}_{\{1,-1,0\}}) = \frac{1 - t_1^4 t_2^2}{(1 - t_1^2 x)(1 - t_1^2/x)(1 - t_1^2 t_2 y)(1 - t_1^2 t_2/y)(1 - t_2^2)}, \quad (5.33)$$

where $x = x_1 x_3$ and $y = x_1 x_2/x_3$ are two new $SU(2)_x \times SU(2)_y$ weights, this group being a particular subgroup of the hidden $SU(2)_1 \times SU(2)_2 \times SU(2)_3$ of the master space of SPP. This moduli space is a complete intersection of dimension 4 generated by the 5 $U(1)_3$ invariants

$$A_1, \quad A_2, \quad M_1 = B_1 C_1, \quad M_2 = C_2 B_2, \quad M_3 = B_1 B_2, \quad (5.34)$$

which satisfy the relation

$$M_1 M_2 = A_1 A_2 M_3. \quad (5.35)$$

The mesonic spectrum can be computed by taking the scaling $t_1 = e^{-\mu b_1}, t_2 = e^{-\mu b_2}, x = 1, y = 1$. The original top holomorphic form on \mathbb{C}^6 scales as the product of weights of the six perfect matchings $t_1^4 t_2^2$. This gives the CY condition $4b_1 + 2b_2 = 2$ and the volume function becomes

$$V(b_1, b_2; \widetilde{SPP}_{\{1,-1,0\}}) = \frac{1}{4b_1^2 b_2 (2b_1 + b_2)}. \quad (5.36)$$

This gives a minimum at $b_1 = b_2 = 1/3$, reproducing the mesonic spectrum of D_3 . Using this result we can rewrite the two $U(1)$ fugacities as $t_1 = tq, t_2 = t/q^2$, where t is the fugacity for the R charge and q is a fugacity for a global charge. The Hilbert series of D_3 takes the form

$$g(t, q, x, y; \widetilde{SPP}_{\{1,-1,0\}}) = \frac{1 - t^6}{(1 - t^2 q^2 x)(1 - t^2 q^2/x)(1 - t^3 y)(1 - t^3/y)(1 - t^2/q^4)}. \quad (5.37)$$

By symmetry the six normalized volumes are equal to $1/3$ and this gives R-charge $1/3$ for the B fields and R-charge $2/3$ for the A and C fields.

5.4.2 (ii) $(k_1, k_2, k_3) = (-2, 1, 1)$

is interesting because it is the simplest case with irrational R-charges. The two points q_i split and we obtain a toric diagram with six external points as in Figure 13.

The four-fold is obtained from the master space by modding out by $U(1)_2 - U(1)_3$ acting on the perfect matchings as $(1, -1, 2, -2, 0, 0)$. By the redefinition $x_1 \rightarrow x_1 w, x_2 = w^2$ this

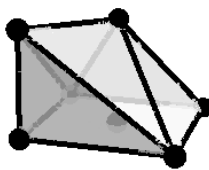


Figure 13. Toric diagram for $\widetilde{SPP}_{\{-2,1,1\}}$.

becomes equivalent to an integration over w .

$$g(t_1, t_2, x_1, x_3; \widetilde{SPP}) = \int \frac{dw}{2\pi i w} \frac{1 - t_1^4}{(1 - t_1^2 x_1 x_3 w)(1 - t_1^2 x_1 w/x_3)(1 - t_1^2 x_3/x_1 w)(1 - t_1^2/x_1 x_3 w)(1 - t_2 w^2)(1 - t_2/w^2)} \quad (5.38)$$

This integral can be easily done. We will not report the long resulting expression but we write the volume functional to be minimized. The original top holomorphic form on \mathbb{C}^6 scales as the product of weights of the six perfect matchings $t_1^4 t_2^2$. In order to have a scaling of 2 we write $t_2 = e^{-b\mu}$ and $t_1 = e^{-(2-2b)\mu/4}$. We can safely put $x_1 = x_3 = 1$ by symmetry and we have,

$$\lim_{\mu \rightarrow 0} \mu^4 g\left(e^{-(1-b)\mu/2}, e^{-\mu b}, 1, 1; \widetilde{SPP}\right) = \frac{16 - 28b + 16b^2 - 3b^3}{2b(1 - 2b + b^2)(16 - 32b + 24b^2 - 8b^3 + b^4)}, \quad (5.39)$$

whose minimum is at

$$b = \frac{1}{18} \left(19 - \frac{37}{(431 - 18\sqrt{417})^{1/3}} - (431 - 18\sqrt{417})^{1/3} \right) \sim 0.319. \quad (5.40)$$

This is the R-charge of the fields called B . The R-charge for A, C is then about 0.681 corresponding to $(1 - b)$.

Symmetry and the Hilbert series have determined uniquely the R-charges of the fields. There is a baryonic symmetry that we can identify with w but, by symmetry, it does not contribute to R-charges. It is nevertheless interesting to look at divisors and volumes. We have six external points in the toric diagram, labeled by the six perfect matchings, corresponding to five-cycles where we can wrap five-branes. Call a_i the R-charge of a brane wrapped on the i -th cycle. The six numbers a_1, \dots, a_6 are in correspondence with $p_1, p_2, p_3, p_4, q_1, q_2$, respectively. We can compute the value of a_i by any of the methods in the appendix. Using the Hilbert series method we have to recompute the Molien integral in (5.32) with the insertion of (the inverse of) the weight of the corresponding perfect matching. For example, after all redefinitions, p_1 has weight $t_1 x_1 w z$ under the four toric symmetries, the baryonic symmetry and the symplectic quotient charge z , so we compute

$$g(D_1; t_1, x_1, x_2, t; \widetilde{SPP}_{\{-2,1,1\}}) = \int \frac{dz}{2\pi i z} \frac{dw}{2\pi i w} \frac{(t_1 x_1 w z)^{-1}}{(1 - t_1 x_1 w z)(1 - t_1 z/x_1 w)(1 - t_1 x_2 w^2/z)(1 - t_1 w^2/x_2 z)(1 - t_3)(1 - t_4)} \quad (5.41)$$

and

$$a_1 = \lim_{\mu \rightarrow 0} \frac{1}{\mu} \left(\frac{g(D_1; e^{-(1-b)\mu/2}, e^{-\mu b}, 1, 1; \widetilde{SPP})}{g(e^{-(1-b)\mu/2}, e^{-\mu b}, 1, 1; \widetilde{SPP})} - 1 \right) \sim 0.305. \quad (5.42)$$

Analogously we compute $a_2 = a_1$, $a_3 = a_4 \sim 0.319$ and $a_5 = a_6 \sim 0.376$. The fields have R-charges that follows from their expression in terms of perfect matchings given in table 3,

$$A_1 \rightarrow a_1 + a_5, A_2 \rightarrow a_2 + a_6, C_1 \rightarrow a_1 + a_6, C_2 \rightarrow a_2 + a_5, B_1 \rightarrow a_3, B_4 \rightarrow a_4, \quad (5.43)$$

and we recover the previous result (5.40). It is interesting that $a_{1,2}$ and $a_{5,6}$ are different. We see that consistently with symmetry, the points are paired two by two, but while $a_{3,4}$ are the R-charge values of the fields $B_{1,2}$, the R-charges $a_{1,2,5,6}$ do not correspond to elementary fields. A and C can be obtained by wrapping branes on pairs of cycles. It remains to understand whether there are consistent objects wrapped on the cycles 1, 2, 5, 6 in the spectrum of the Chern-Simons theory.

5.4.3 (iii) $(k_1, k_2, k_3) = (0, -1, 1)$

was already worked out in detail in [14, 30]. For completeness we briefly discuss this case as well. The choice $(k_1, k_2, k_3) = (0, -1, 1)$ gives the toric diagram of $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}^2$. It is obtained by dividing the master space by the gauge group without CS term, $U(1)_1$. As seen from the previous assignment of charges, this is equivalent to integrate over x_3 . The four fugacities t_1, t_2, x_1, x_2 correspond to the four toric actions of the Calabi-Yau four-fold. This computation is by now familiar and we obtain

$$g(t_i, x_1; \widetilde{SPP}_{0,-1,1}) = \frac{1 + t_1^4}{(1 - t_1^4 x_1^2)(1 - t_1^4/x_1^2)(1 - t_2 x_2)(1 - t_2/x_2)}. \quad (5.44)$$

Thus, we find indeed the Hilbert series⁸ for $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}^2$.

By symmetry, we expect the same R-charge for the fields A, B, C which should be $1/2$ to fit with the superpotential. In fact, the minimization of the volume functional reproduces this result. There is a baryonic charge but its value is by symmetry zero. The four external perfect matchings p_1, \dots, p_4 in the 3d toric diagram carry an R-charge a_i that can be computed as in the previous subsection, but with no surprises this time: $a_i = 1/2$, consistent with our previous discussion.

5.5 The $\widetilde{\mathbb{C}^3/\mathbb{Z}_3}$ theories

This set of models are given by the tiling and quiver of Figure 14. There are three groups and three sets of chiral fields U_i, V_i, W_i , $i = 1, 2, 3$ transforming in the $(N, \bar{N}, 0)$, $(0, N, \bar{N})$ and $(\bar{N}, 0, N)$ representation of the gauge group, respectively, and interacting with the superpotential

$$W = \epsilon_{ijk} U_i V_j W_k. \quad (5.45)$$

⁸As already discussed in [14, 30], the result can be easily recovered by looking at independent invariants under the $U(1)_1$ action: $M_{11} = A_1 A_2, M_{21} = C_1 A_1, M_{12} = A_2 C_2, M_{22} = C_1 C_2, B_1, B_2$ which satisfy the equations $M_{12} M_{21} = M_{11} M_{22} = M_{11}^2$.

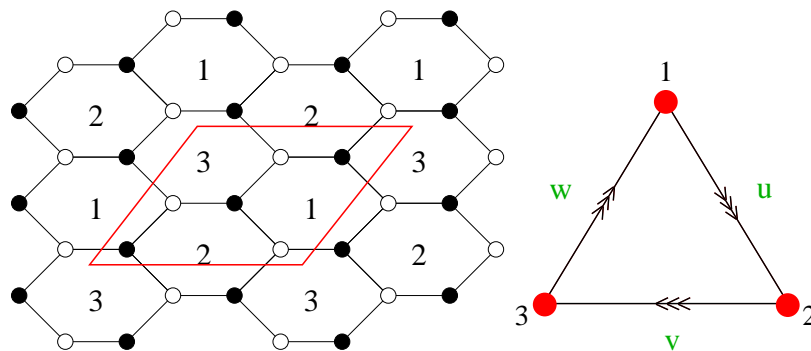


Figure 14. (i) Brane tiling for $\widetilde{\mathbb{C}^3/\mathbb{Z}_3}$ (ii) The corresponding quiver

The levels of the groups can be chosen to be k_1, k_2 and $-k_1 - k_2$, respectively. These models are special since they appear to be the simplest chiral (in the 3+1 dimensional sense) models that exhibit a spectrum of non-trivial anomalous dimensions. The superpotential is cubic and therefore allows for scaling dimensions which are far from the canonical scaling in 2+1 dimensions.

The Kasteleyn matrix is

$$K = \begin{pmatrix} z^{-k_1+k_2} & z^{-k_1-2k_2}x & y^{-1}z^{2k_1+k_2} \\ z^{-k_1-2k_2} & z^{2k_1+k_2} & z^{-k_1+k_2} \\ z^{2k_1+k_2}y & z^{-k_1+k_2} & x^{-1}z^{-k_1-2k_2} \end{pmatrix}, \quad (5.46)$$

and the permanent is

$$\text{perm } K = z^{-3k_1-6k_2} + z^{-3k_1+3k_2} + z^{6k_1+3k_2} + x^{-1} + y^{-1} + xy. \quad (5.47)$$

After an appropriate rescaling, this gives a toric diagram with the following points:

$$(-1, 0, 0), (0, -1, 0), (1, 1, 0), (0, 0, -k_1 - 2k_2), (0, 0, -k_1 + k_2), (0, 0, 2k_1 + k_2).$$

These models are identified with the two parameter set of theories,⁹ $Y^{p,k}(\mathbb{C}P^2)$, with $p = k_1 + k_2, k = 2k_1 + k_2$ for $k_1, k_2 \geq 0$. In the following, we will look at special cases.

5.5.1 The cone over $M^{1,1,1}$

The case of CS parameters $k_1 = k_2 = 1$ deserves special attention. The corresponding four-fold is the cone over the coset manifold $M^{1,1,1} = \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)/\text{SU}(2) \times \text{U}(1) \times \text{U}(1)$ [58]¹⁰ with global symmetry $\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)_R$. The coincidence of this global symmetry with the gauge group of the standard model for particle interactions was a reason for enhanced activity back in the 80's. An attempt to give a Yang-Mills theory dual can be found in [9]. Here we focus on the Chern-Simons dual theory.

We can use the refined Kasteleyn matrix to learn about fields and perfect matchings

$$K = \begin{pmatrix} U_1 z^{-k_1+k_2} & V_2 z^{-k_1-2k_2}x & W_3 y^{-1}z^{2k_1+k_2} \\ V_3 z^{-k_1-2k_2} & W_1 z^{2k_1+k_2} & U_2 z^{-k_1+k_2} \\ W_2 z^{2k_1+k_2}y & U_3 z^{-k_1+k_2} & V_1 x^{-1}z^{-k_1-2k_2} \end{pmatrix},$$

⁹See [31] and the revised version of [57].

¹⁰This is called $M^{3,2}$ in [57].

$$\begin{aligned} \text{perm } K &= z^{-3k_1-6k_2} V_1 V_2 V_3 + z^{-3k_1+3k_2} U_1 U_2 U_3 + z^{6k_1+3k_2} W_1 W_2 W_3 + x^{-1} U_1 V_1 W_1 \\ &+ y^{-1} U_3 V_3 W_3 + xy U_2 V_2 W_2. \end{aligned} \tag{5.48}$$

As we see, there are six perfect matchings $q_2, q_1, q_3, p_1, p_3, p_2$ corresponding with the monomial in $\text{perm}K$ with the given order. From the 3+1 dimensional perspective, the three p_i are external perfect matchings and the three q_i are associated with the internal point of multiplicity 3. From the 2+1 dimensional perspective, q_2 and q_3 become new external points while q_1 remains an internal point. The five external perfect matchings $p_i, q_{2,3}$ can be set in correspondence with the five external points in the toric diagram and the corresponding five divisors D_i . From the Kasteleyn matrix we see that U_1 belongs to the perfect matchings p_1 and q_1 so it can be parametrized as the product $p_1 q_1$. in a similar fashion we read the parameterization of the other fields in terms of perfect matchings,

$$U_i = p_i q_1, \quad V_i = p_i q_2, \quad W_i = p_i q_3 \tag{5.49}$$

The moduli space is given by modding out the five dimensional master space by the $U(1)$ gauge symmetry prescribed by the CS terms. Let us describe both the master space and the resulting four-dimensional Calabi-Yau.

The master space is given by the perfect matchings modulo relations. There is one relation among perfect matching $p_1 + p_2 + p_3 = q_1 + q_2 + q_3$ and this gives the, by now usual [32], description of the master space as $\mathbb{C}^6 / \{-1, -1, -1, 1, 1, 1\}$, where we order the perfect matchings as $p_1, p_2, p_3, q_1, q_2, q_3$. The master space is a five dimensional toric variety with $SU(3) \times SU(3) \times U(1)$ symmetry, where the second $SU(3)$ is hidden from a 3+1 dimensional perspective. We can introduce weights for the action of the global symmetry on perfect matchings as follows:

$$p_1, p_2, p_3, q_1, q_2, q_3 \rightarrow ty, tx, t/xy, x_1, x_2, 1/x_2 x_1 \tag{5.50}$$

where t is the $U(1)$ charge, x, y are weights for the first $SU(3)$ and x_1, x_2 are weights for the second $SU(3)$. As seen from formula (5.49) x_1 and x_2 correspond to the two independent charges under the gauge group. The Hilbert series for the master space is

$$\int \frac{dz}{2\pi iz} \frac{1}{(1 - ty/z)(1 - tx/z)(1 - txy/z)(1 - x_1 z)(1 - zx_2)(1 - z/x_2 x_1)} \tag{5.51}$$

The Calabi Yau four-fold is obtained now by modding by the gauge group $U(1)_1 - U(1)_2$ which, as seen from equation (5.49) corresponds to the action $\{0, 0, 0, 2, -1, -1\}$ on perfect matchings and it breaks the global symmetry to $SU(3) \times SU(2) \times U(1)$. Note that part of the hidden symmetry now becomes a symmetry of the mesonic moduli space in the 2+1 dimensional theory. By redefining $x_1 = w^2, x_2 = \tilde{x}/w$, this just corresponds to integrating over w . We can use t, x, y, \tilde{x} to parametrize the four toric symmetries of the Calabi-Yau. \tilde{x} is now interpreted as an $SU(2)$ weight. The Hilbert series for the mesonic moduli space depends on t, x, y, \tilde{x} and is given by

$$g(t, x, y, \tilde{x}; \widetilde{\mathbb{C}^3/Z_3}) = \int \frac{dz}{2\pi iz} \frac{dw}{2\pi i} \frac{1}{((1 - ty/z)(1 - tx/z)(1 - txy/z)(1 - w^2 z))} \times$$

$$\begin{aligned}
 & \times \frac{1}{(1 - z\tilde{x}/w)(1 - z/w\tilde{x})} \\
 & = \sum_{k=0}^{\infty} [3k, 0; 2k] t^{3k} \tag{5.52}
 \end{aligned}$$

where $[n, m; s]$ denotes irreps of $SU(3) \times SU(2)$. From the last expression we recognize indeed the KK spectrum of M theory compactified on $M^{1,1,1}$ [9].

We can extract the volume formula from the Hilbert series expression and minimize it. However we know the result without need of computation. The $SU(3) \times SU(2)$ symmetry immediately gives $x = y = \tilde{x} = 1$. Moreover the Calabi-Yau top form scale as $\prod p_i q_i \sim t^3$ and therefore t corresponds to a dimension $2/3$. The see that the KK spectrum consists of multiplets of dimensions $2k, k = 0, 1, 2, \dots$ as indeed known from supergravity [9, 59].

The full moduli space, including baryonic operators, has an extra charge with fugacity w as above. The mesonic operators are independent of w but the R-charges of the elementary fields depend on w . Since $SU(2)$ exchanges V, W , they have same R-charge but this can be different from the R-charge of U . We can use the external perfect matchings to parametrize the five global charges. Introducing R-charges a_1, a_2, a_3, a_4, a_5 associated with p_1, p_2, p_3, q_2, q_3 , the fields have R-charge:

$$U_i \rightarrow a_i, \quad V_i \rightarrow a_i + a_4, \quad W_i \rightarrow a_i + a_5 \quad i = 1, 2, 3 \tag{5.53}$$

It is tempting to compute the numbers a_i with the same rule as in 3+1 dimensions: a_i is the R-charge corresponding to a brane wrapped on the five-cycle corresponding to the i -th external point of the toric diagram and it is computed as a normalized volume.

The volumes were computed in [9] and reobtained in [31]. Divisors $D_{1,2,3}$ correspond to a normalized volume $4/9$ and divisors $D_{4,5}$ to $1/3$. We can recheck it with the method discussed in the appendix which indeed gives, for example,

$$\begin{aligned}
 g(D_1, ; t, x, y, \tilde{x}; \widetilde{\mathbb{C}^3/Z_3}) &= \int \frac{dz}{2\pi iz} \frac{dw}{2\pi iw} \frac{(ty/z)^{-1}}{((1 - ty/z)(1 - tx/z)(1 - txy))} \times \\
 & \times \frac{1}{(1 - w^2 z)(1 - z\tilde{x}/w)(1 - z/w\tilde{x})} \tag{5.54}
 \end{aligned}$$

and

$$a_1 = \lim_{\mu \rightarrow 0} \frac{1}{\mu} \left(\frac{g(D_1; e^{-2\mu/3}, 1, 1, 1; \widetilde{\mathbb{C}^3/Z_3})}{g(e^{-2\mu/3}, 1, 1, 1; \widetilde{\mathbb{C}^3/Z_3})} - 1 \right) = \frac{4}{9} \tag{5.55}$$

and similarly for the other divisors.

We get $a_1 = a_2 = a_3 = 4/9$ and $a_4 = a_5 = 1/3$. We see from (5.49) that we expect R-charge $4/9$ for U and $7/9$ for V, W. This is consistent with the CS theory lagrangian. As in the other examples in this paper we see that no elementary field is associated with $D_{4,5}$ and the value $1/3$. It remains an open question to understand whether there are really states in the spectrum obtained from M5 branes wrapped on the base of D_4 and D_5 and what is their description in the CS theory.

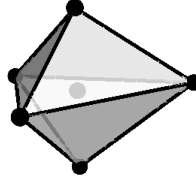


Figure 15. Toric diagram for the cone over $M^{1,1,1}$.

5.5.2 The case of $\widetilde{\mathbb{C}^3/\mathbb{Z}_3\{1,0,-1\}}$

This case corresponds to the manifold $Y^{12}(\mathbb{C}P_2)$ [57]. All the points q_i now become external points. There are altogether five external points because q_1 and q_2 coincide.

The Calabi Yau four-fold is now obtained by modding out the master space by the gauge group $U(1)_2$ with action $\{0, 0, 0, -1, 1, 0\}$ on the perfect matchings. This breaks the global symmetry to $SU(3) \times U(1)^2$. After the redefinition $x_1 = 1/w, x_2 = w/q$ it just corresponds to integrate over w . We can still use t, x, y, q to parametrize the four toric symmetries of the Calabi-Yau. The Hilbert series for the mesonic moduli space depends on t, x, y, q and is given by

$$g(t, x, y, q; \widetilde{\mathbb{C}^3/\mathbb{Z}_3\{1,0,-1\}}) = \int \frac{dz}{2\pi iz} \frac{dw}{2\pi iw} \frac{1}{((1 - ty/z)(1 - tx/z)(1 - txy/z))} \times \quad (5.56)$$

$$\times \frac{1}{(1 - z/w)(1 - zw/q)(1 - zq)}$$

The explicit expression is too long and we report it only in the case $x = y = q = 1$,

$$g(t, x = 1, y = 1, q = 1; \widetilde{\mathbb{C}^3/\mathbb{Z}_3\{1,0,-1\}}) = \frac{1 + 2t + 6t^2 + 2t^3 + t^4}{(1 - t)(1 - t^2)^3}. \quad (5.57)$$

We can extract the volume formula from the Hilbert series expression and minimize it. From the scaling of the top holomorphic form as t^3 we still have that the charge for t is $2/3$. The $SU(3)$ symmetry immediately gives $x = y = 1$ and we obtain the volume function,

$$\lim_{\mu \rightarrow 0} \mu^4 g\left(e^{-2\mu/3}, 1, 1, e^{-\mu b_q}; \widetilde{\mathbb{C}^3/\mathbb{Z}_3\{1,-1,0\}}\right) = \frac{243(16 + 12b_q + 9b_q^2)}{(8 + 6b_q - 9b_q^2)^3} \quad (5.58)$$

whose minimization gives

$$b_q = \frac{1}{6} \left((181 + 24\sqrt{78})^{1/3} - \frac{23}{(181 + 24\sqrt{78})^{1/3}} - 3 \right) \sim 0.197 \quad (5.59)$$

The full moduli space, including baryonic operators, has an extra charge with fugacity q as above. We use again the external perfect matchings to parametrize the five global charges. This time we introduce $a_1, a_2, a_3, a_4/2, a_5$ associated with $p_1, p_2, p_3, q_1 = q_2, q_3$, so that the fields have R-charge:

$$U_i \rightarrow a_i + a_4/2, \quad V_i \rightarrow a_i + a_4/2, \quad W_i \rightarrow a_i + a_5 \quad i = 1, 2, 3 \quad (5.60)$$

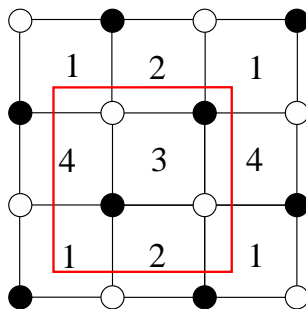


Figure 16. Brane tiling for \mathbb{F}_0^I .

The values of a_i can be computed as in the previous examples, obtaining rational expressions in b_q too long to be reported here. The only subtlety, compared with previous cases, is that we have an external point with multiplicity two: the corresponding R-charge has to be divided among the corresponding fields.¹¹ The numerical value is:

$$a_1 = a_2 = a_3 = 0.451, \quad a_4 = 0.235 \quad a_5 = 0.414 \quad (5.61)$$

The result seems to agree with the numerical computation in [57] using the explicit metric.

We see from (5.49) that we expect R-charge 0.569 for U, V and 0.865 for W . This is consistent with the CS theory lagrangian. As in the other examples we see that no elementary field is associated with $D_{4,5}$ and the values a_4, a_5 .

5.6 The $\tilde{\mathbb{F}}_0$ theories

\mathbb{F}_0 has two toric phases (in 3+1 dimensions) [60]. Their tilings are depicted in Figure 16, where we denote the model as \mathbb{F}_0^I , and in Figure 18 where the model is denoted by \mathbb{F}_0^{II} . (This order of notation is chosen based on the complexity of the models. For instance, the number of fields is 8 in \mathbb{F}_0^I and 12 in \mathbb{F}_0^{II} .) A choice of 2+1 dimensional CS levels does not in general commute with 3+1 dimensional toric duality. Therefore, we need to treat each model separately: they give rise to their own set of 2+1 dimensional field theories. It is essential to investigate the relation between toric duality and the choice of CS levels but this will not be done in the present paper (see however some speculations about a possible dual ABJM theory in appendix B). We proceed as in the previous examples.

The levels of the groups are chosen to be k_1, k_2, k_3 and $k_4 = -k_1 - k_2 - k_3$, respectively. We write the Kasteleyn matrix,

$$K = \begin{pmatrix} z^{k_3-k_2} + z^{k_1-k_4}x & z^{-k_3+k_4} + \frac{z^{-k_1+k_2}}{y} \\ z^{-k_3+k_4} + z^{-k_1+k_2}y & z^{k_3-k_2} + \frac{z^{k_1-k_4}}{x} \end{pmatrix}, \quad (5.62)$$

and find that after substituting $k_4 = -k_1 - k_2 - k_3$, the permanent is

$$\text{perm } K = z^{-2k_1-2k_2-4k_3} + z^{-2k_2+2k_3} + z^{-2k_1+2k_2} + z^{4k_1+2k_2+2k_3} \quad (5.63)$$

¹¹A similar ambiguity appears in the computation of a_4 using the Hilbert series for the line bundle D_4 : we need to use the perfect matching $q_1 + q_2$.

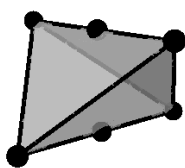


Figure 17. 3d toric diagram for $(\mathbb{C}^2/\mathbb{Z}_2)^2$.

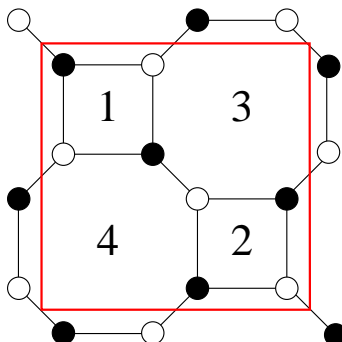


Figure 18. Brane tiling for \mathbb{F}_0^{II} .

$$+\frac{z^{2k_1+2k_3}}{x} + z^{2k_1+2k_3}x + \frac{z^{-2k_1-2k_3}}{y} + z^{-2k_1-2k_3}y.$$

To disentangle this expression let us define linear combinations of levels, $a = k_1 + k_3$, $b = k_2 + k_3$ and $c = k_1 + k_2$, and write

$$\text{perm } K = \left(x + \frac{1}{x}\right) z^a + \left(y + \frac{1}{y}\right) z^{-a} + z^{-a-b} + z^{-a+b} + z^{a-c} + z^{a+c}. \quad (5.64)$$

This gives a three-parameter set of toric moduli spaces.

By setting $a = 1$ and $b = c = 0$ (that is $k_1 = -k_2 = k_3$), we get $(\mathbb{C}^2/\mathbb{Z}_2)^2$ as the moduli space for the theory $\tilde{\mathbb{F}}_{0\{1,-1,1,-1\}}^I$. A similar analysis to the previous examples can be done for scaling exponents and Hilbert series but we will skip this.

5.6.1 The \mathbb{F}_0^{II} tiling and its family of theories

For model \mathbb{F}_0^{II} we will be very brief. Set the levels of the groups as k_1, k_2, k_3 and $k_4 = -k_1 - k_2 - k_3$, respectively.

The resulting Kasteleyn matrix and permanent are

$$K = \begin{pmatrix} z^{-k_1+k_4} & z^{k_1-k_3} & 0 & \frac{z^{k_3-k_4}}{xy} \\ z^{k_1-k_3} & z^{-k_1+k_4} & z^{k_3-k_4} & 0 \\ 0 & z^{k_3-k_4}x & z^{k_2-k_3} & z^{-k_2+k_4} \\ z^{k_3-k_4}y & 0 & z^{-k_2+k_4} & z^{k_2-k_3} \end{pmatrix}$$

$$t^{4k_3} \cdot \text{perm } K = t^{-4k_1} + t^{-4k_2} + t^{-6(k_1+k_2)} + t^{2(k_1+k_2)} + t^{4(k_1+k_2+3k_3)} \quad (5.65)$$

$$+ t^{2(k_2+3k_3)} \left(\frac{1}{x} + x\right) + t^{2(k_1+3k_3)} \left(\frac{1}{y} + y\right)$$

In the case of $k_1 = k_2$, where the nodes that are external points in the 2d toric diagram are still coplanar, the toric diagram obtained from the above permanent seems to match a subset of diagrams of [31] for $Y^{p,k}(\mathbb{C}P^1 \times \mathbb{C}P^1)$.

6 Conclusions

In this paper we study in detail the properties of 2+1 dimensional Chern-Simons theories with an abelian moduli space that is a Calabi-Yau four-fold following the construction in [30]. In particular, we describe an extension of the “fast forward algorithm” of [25] to efficiently determine the toric data of the Calabi-Yau starting from the tiling.

Such theories stand naturally as candidates for the world-volume theory of membranes probing the Sasaki-Einstein base of the Calabi-Yau and they may become an important ingredient in our understanding of the AdS_4/CFT_3 correspondence. In order to check the duality between the Chern-Simons theory and the $AdS_4 \times H$ background, we study the spectrum of mesonic and baryonic excitations, as predicted by the supergravity dual. Unfortunately, it is still difficult to have exact quantum field theory results in 2+1 dimensions, so we can only perform some consistency checks on our theories. As expected, the mesonic spectrum of a Chern-Simons theory agrees with the KK spectrum of the compactification on H . This is not a surprise: as is well known from similar analysis in 3+1 dimensions, this is a consequence of the relation between holomorphic functions on the Calabi-Yau and eigenvectors of the Laplacian on H . The R-charges that we obtain by minimization are consistent in all our examples with the superpotential structure of the Chern-Simons Lagrangian and satisfy all the relevant unitary bounds. We point out a puzzle regarding the baryonic spectrum, where it seems that five-branes wrapped on certain cycles do not correspond to baryons made with elementary fields. This remains a problem to be solved for a proper understanding of the proposed duality.

The class of Chern-Simons theories obtained by tilings probably does not exhaust all the theories dual to Calabi-Yau four-folds, not even the toric ones. The analysis of the moduli space is not limited to theories coming from tilings and could be applied to more general theories, obtained for example from crystals [27, 28]. This is left for future work.

Acknowledgments

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A Prescriptions for computing the Hilbert series

Consider the description of the Calabi-Yau four-fold X as a symplectic quotient on the space of perfect matchings. We usually want to write refined Hilbert series depending on

a set of global charges characterizing the four-fold. In particular we always introduce at least four weights corresponding to the four toric $U(1)$ actions on X .

The perfect matchings are indicated as p_α where the index α runs from one to the number c of integer points in the 2d toric diagram, including multiplicities. The number c can be large and depends in a non trivial way on the form of the tiling. The perfect matchings are subsets of the collections of edges (elementary fields) and we can form formal linear combinations with integer coefficients $\sum n_\alpha p_\alpha$, $n_\alpha \in \mathbb{Z}$. Please notice that we will use an additive notation for perfect matchings, when regarded as collections of fields. The p_α , as collections of edges in the tiling, will in general satisfy some linear relations $\sum_\alpha Q_\alpha p_\alpha = 0$ that translate into vectors of charges for a symplectic quotient. The coherent component of the master space, which is the biggest irreducible component of the F-term solutions, has dimension $g + 2$ and it is then obtained by modding the space of perfect matchings \mathbb{C}^c by the charge vectors corresponding to the $c - g - 2$ linear relations satisfied by the perfect matchings. Another way to say this is that the solution of the F-terms can be written as

$$X_i = \prod_{\alpha=1}^c p_\alpha^{P_{i\alpha}}, \tag{A.1}$$

where the element of the matrix $P_{i\alpha}$ is 1 if the field X_i belongs to the perfect matching p_α and 0 if it does not. The redundancy in the parametrization (A.1) is given by the kernel of the matrix P and one can prove that the vectors of charges Q span precisely the kernel of P . All the global and gauge symmetries of the CS theory act on the solutions of F-terms and can be lifted to an action on p_α . We can then give weights under global and gauge actions to the p_α .

We deal in the text with many different symplectic quotients of \mathbb{C}^c :

- By modding by the \mathbb{C}^* action corresponding to linear relations among perfect matchings, we obtain the master space.
- By adding the $g - 2$ gauge charges we obtain the CY four-fold X .
- By adding the last gauge charge that is reduced to discrete symmetry by the presence of the CS terms we obtain the CY three-fold associated with the tiling.

The Hilbert series for all these cases can be computed with the integral Molien formula which schematically reads

$$g(t_i; X) = \int \prod_{i=1}^G \frac{dz_i}{2\pi i z_i} \frac{1}{\prod_{\alpha=1}^c (1 - t_\alpha Z_\alpha)} \tag{A.2}$$

where G is the total number of \mathbb{C}^* actions we are modding out. In this formula the dummy variable t_α correspond to the perfect matching p_α and it is the weight of the perfect matching itself under the global symmetry of the theory. Z_α denotes the monomial weight of the α -th perfect matching p_α in terms of the z_α . For example, if we want the Hilbert series of the Calabi-Yau four-fold X , t_α will denote the weight under the four global charges

of the toric action on X and Z_α is the weight under the a total of $c - 4$ charges, divided into $c - g - 2$ charges coming from the linear relations among perfect matchings and the $g - 2$ D-term charges. Notice that we can assign c different weights t_α to the perfect matchings, but, due to the $c - 4$ integrations, only four of them will be independent. Similar arguments apply to the computation of the Hilbert series for the master space or for the Calabi-Yau three-fold associated with the tiling.

Sometimes the description in terms of master space, although conceptually crystal clear and directly connected to the field theory, can be cumbersome because of the many contour integrations to be performed, especially when c is large. We can then resort to different types of descriptions. Any toric variety can be written, for example, as a symplectic quotient in \mathbb{C}^d where d is the number of external points in the toric diagram [49]. The information about the 3d toric diagram comes to us from the magnetic Kasteleyn matrix and allows for computation using a Molien integral on d variables and $d - 4$ contour integrals. This description is useful when $d \ll c$. Finally, in some lucky cases, we will be able to write our variety, or the master space, as a set of algebraic equations which defines a complete intersection. The computation of the Hilbert series is then straightforward due to this property.

To study volumes, we can take two different approaches. Following [61], we are led to investigate divisors on the CY and their associated line bundles. Recall that there is a correspondence between external points of the toric diagram and divisors D_a . We call external perfect matchings those corresponding to external points in the toric diagram. The Hilbert series for the line bundle associated with D_a is obtained as follows. Given a divisor D_a corresponding to an external perfect matching p_a , we can conjecture the following modified Molien formula for the Hilbert series of holomorphic sections of D_a

$$g(D_a; X) = \int \prod_{i=1}^G \frac{dz_i}{2\pi i z_i} \frac{(t_a Z_a)^{-1}}{\prod_{\alpha=1}^c (1 - t_\alpha Z_\alpha)} \tag{A.3}$$

The rationale for this formula comes from the fact that every line bundle on the CY four-fold should come from the ambient space \mathbb{C}^c . A line bundle on \mathbb{C}^c is necessarily free, but we can obtain line bundle on the CY by letting the charges z_i to act on the fiber.

As discussed in section 4 we can extract the volume of the base of X from the leading pole of the Hilbert series for X . We can analogously extract the volumes of the base of D_a from the Hilbert series for holomorphic sections of the line bundle [61]. All these results will depend on a Reeb vector $b = (b_1, b_2, b_3, b_4)$, specifying a linear combination of the four toric actions, restricted by the only condition that it gives charges 2 to the holomorphic top form. The actual value of b is obtained by minimizing the volume of the base of X . Knowing b at the minimum we can compute volumes of the seven manifold X and its five-cycles.

Since we are interested in the R-charge of a brane wrapped on the base of the divisor, we need to normalize the volumes according to formula (4.5). The following nice formula works both in 3+1 dimensions and in 2+1 dimensions,

$$\frac{g(D_i, e^{-\mu b_i}; X)}{g(e^{-\mu b_i}; X)} \sim 1 + \Delta(b_i)\mu + \dots \tag{A.4}$$

Here b is the Reeb vector whose value is obtained by minimizing the coefficient of the leading pole of the Hilbert series $g(t_i; X)$ for $\mu \rightarrow 0$ as discussed in the main text. We point out that the previous formulae stand as conjectures tested in many interesting cases against known results. In particular they should be used with care in cases where the Calabi-Yau has singularities or multiplicities on the external points.

By re-elaborating the results in [36] we can also write an explicit formula in terms of the toric data. Given an external point v_a of the 2d toric diagram, with associated divisors D_a , consider the clockwise ordered sequence of vectors $w_k, k = 1, \dots, n_a$ in the toric diagram that are adjacent to v_a . Define

$$F_a = \sum_{k=2}^{n_a-1} \frac{(v_i, w_{k-1}, w_k, w_{k+1})(v_i, w_k, w_1, w_{n_a})}{(v_i, b, w_k, w_{k+1})(v_i, b, w_{k-1}, w_k)(v_i, b, w_1, w_{n_a})} \tag{A.5}$$

where (V_1, V_2, V_3, V_4) denotes the determinant of four vectors $V_{1,2,3,4}$. The expression for the R-charge of a five-brane wrapped on the base of D_a is then

$$R_a = \frac{2F_a}{\sum_a F_a} \tag{A.6}$$

These formulae depend on the Reeb vector b , whose value can be found by minimizing the volume functional which in the new setting is given by $\sum_a F_a$. b is always constrained by the request that the holomorphic top form scales appropriately; in the approach with toric data and in all our examples where the CY condition is enforced by taking vectors v_a with fourth coordinates equal to 1, this means $b_4 = 4$. Due to different notations, the vector b entering in these equations is in general related to the vector b entering the Molien formula by a change of basis.

In this approach it is easy to see the important fact that the normalized volumes always add up to 2.

We now give an explicit example of the use of the previous formula. Many other examples are scattered along the paper.

A.1 The cone over $Q^{1,1,1}$

We use an example where all about Hilbert series and volumes is under control, the cone over the manifold $Q^{1,1,1} = \text{SU}(2)^3/\text{U}(1) \times \text{U}(1)$ with symmetry $\text{SU}(2)^3 \times \text{U}(1)_R$. It is not clear at the moment how to construct a Chern-Simons theory with non-abelian moduli space the symmetric product of copies of $C(Q^{1,1,1})$.¹² Nevertheless, it makes perfect sense to analyze the Hilbert series and the volume of divisors as a check of our general formulas. Most of the results could be predicted by using symmetries only, but the purpose of the following discussion is to explain how to use the Hilbert series and the toric data to determine the Reeb vector and volumes.

The nice thing is that the cone over $Q^{1,1,1}$ has a symplectic quotient description: $C(Q^{1,1,1})$ is \mathbb{C}^6 modded by two \mathbb{C}^* with charge vectors $\{1, 1, -1, -1, 0, 0\}, \{0, 0, 1, 1, -1, -1\}$. The toric diagram has indeed six external points satisfying two linear relations given by the previous vectors.

¹²For Yang-Mills approaches see [9, 12].

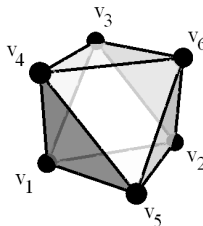


Figure 19. Toric diagram for $Q^{1,1,1}$.

Introduce weights $t_1 = tx_1, t_2 = t/x_1, t_3 = tx_2, t_4 = t/x_2, t_5 = tx_3, t_6 = t/x_3$ for the 6 external points, where x_1, x_2, x_3 are weights for $SU(2)^3$ and t is a weight for $U(1)_R$. The Hilbert series is

$$\begin{aligned}
 g(t_i; C(Q^{1,1,1})) &= \int \frac{dz_1}{2\pi iz_1} \frac{dz_2}{2\pi iz_2} \frac{1}{(1-t_1 z_1)(1-t_2 z_1)(1-t_3 z_2/z_1)} \times \\
 &\quad \times \frac{1}{(1-t_4 z_2/z_1)(1-t_5/z_2)(1-t_6/z_2)} \\
 &= \sum_{n=0}^{\infty} [n; n; n] t^{3n}, \tag{A.7}
 \end{aligned}$$

where $[n; n; n]$ denotes the representation of dimension $(n+1)^3$ of $SU(2)^3$. We recognize the familiar KK spectrum of chiral multiplets in the compactification on $Q^{1,1,1}$, which indeed transform in the $[n; n; n]$ representation [59].

As an aside we note that explicit expressions for the characters of $SU(2)$ are simple and take the form,

$$[n_1; n_2; n_3] = \prod_{i=1}^3 \frac{x_i^{n_i+1} - x_i^{-n_i-1}}{x_i - x_i^{-1}}. \tag{A.8}$$

Using this expression the Hilbert series takes an alternative palindromic form

$$\begin{aligned}
 g(t, x_1, x_2, x_3; C(Q^{1,1,1})) &\tag{A.9} \\
 &= (1 + t^{18} - ([2; 0; 0] + [0; 2; 0] + [0; 0; 2]) (t^6 + t^{12}) + 2[1; 1; 1]t^9) PE [[1; 1; 1] t^3],
 \end{aligned}$$

where this expression indicates that the generators of $C(Q^{1,1,1})$ transform in $[1; 1; 1]$ of $SU(2)^3$.

The volume function is obtained as the leading pole of the Hilbert series for $t_i \rightarrow 1$. We define $t = e^{-\mu b}, x_1 = e^{-\mu b_1}, x_2 = e^{-\mu b_2}, x_3 = e^{-\mu b_3}$ and take the $\mu \rightarrow 0$ limit. The four parameters b, b_1, b_2, b_3 are not independent. They are restricted by the request that the holomorphic top form scales with charge 2. Since holomorphic top form transform as $t_1 \cdots t_6 = t^6$ we learn that $b = 1/3$. The volume functional is then

$$\lim_{\mu \rightarrow 0} \mu^4 g(e^{-\mu/3}, e^{-\mu b_1}, e^{-\mu b_2}, e^{-\mu b_3}; C(Q^{1,1,1})) \tag{A.10}$$

The result is a rational function of b_1, b_2, b_3 that should be minimized. Minimization of a function of three variables can be a non trivial task. In the case of $C(Q^{1,1,1})$ we do not really

need to minimize because we know by symmetry that the result will be $b_1 = b_2 = b_3 = 0$. We leave to the skeptic reader the evaluation and minimization of the previous quantity.

For the same reason, all divisors have the same volume. We sketch the computation of the normalized volume with two different methods. With the first method, we compute the Hilbert series that counts holomorphic sections of a line bundle. By symmetry we can choose any divisor. Let us choose D_1 associated with the external point with weight t_1 . We need to compute the Molien integral with the insertion of the (inverse of) the weight of the external point under all charges. The point 1 has weight $t_1 = tx_1$ and charge z_1 under the two symplectic actions. We thus have, according to (A.3),

$$g(D_1; t, x_i; C(Q^{1,1,1})) \tag{A.11}$$

$$= \oint \frac{dz_1}{2\pi iz_1} \frac{dz_2}{2\pi iz_2} \frac{(t_1 z_1)^{-1}}{(1-t_1 z_1)(1-t_2 z_1)(1-t_3 z_2/z_1)(1-t_4 z_2/z_1)(1-t_5/z_2)(1-t_6/z_2)}$$

Notice that the insertion is just the first monomial appearing in the denominator. Finally, using formula (A.4) for the normalized volume we find

$$\lim_{\mu \rightarrow 0} \frac{1}{\mu} \left(\frac{g(D_1; e^{-\mu/3}, 1, 1, 1; C(Q^{1,1,1}))}{g(D_1; e^{-\mu/3}, 1, 1, 1; C(Q^{1,1,1}))} - 1 \right) = \frac{1}{3} \tag{A.12}$$

which is indeed the right result for volumes in $Q^{1,1,1}$ [9]. Obviously, due to the high symmetry of the problem the outcome is just the scaling dimension of the only non-trivial variable, t . We see in the main text many non-trivial applications of this method.

The second method uses the toric data and formula (A.5). The toric vectors for $C(Q^{1,1,1})$ are

$$\begin{aligned} v_1 &= (1, 0, 0, 1) & v_2 &= (0, 1, 0, 1) & v_3 &= (0, 0, 1, 1) \\ v_4 &= (1, 0, 1, 1) & v_5 &= (1, 1, 0, 1) & v_6 &= (0, 1, 1, 1) \end{aligned} \tag{A.13}$$

We need to compute a volume, as a function of b , for each external point v_a . These volumes are proportional to the quantities F_a given in (A.5). Each F_a is a sum over contributions coming from the points connected to v_a by an edge. We need to specify an order to use during the computation: the points are ordered clockwise as seen from the point v_a . For example, we see from figure 19 that the point 1 is connected to the points 5, 2, 3, 4 with this particular order. We then apply formula (A.5) with $v_1 = (1, 0, 0, 1)$ and the string of points $w = \{v_5, v_2, v_3, v_4\}$ obtaining

$$F_1 = \frac{b_2 + b_3}{b_2 b_3 (b_1 - b_4)(b_1 + b_2 + b_3 - b_4)} \tag{A.14}$$

and similar expressions for the other F_a . The volume functional is $Z = F_1 + F_2 + F_3 + F_4 + F_5 + F_6$ which should be minimized with the constraint that $b_4 = 4$. In this example, by symmetry of the toric diagram, we can predict that $b_1 = b_2 = b_3 \equiv B$ at the minimum and we obtain the functional

$$Z(B) = \frac{24}{B(B-4)(32-36B+9B^2)} \tag{A.15}$$

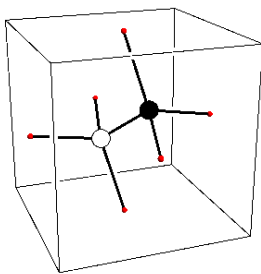


Figure 20. Brane crystal for \mathbb{C}^4 .

whose minimum is at $B = 2$. We then have

$$\frac{2F_a}{\sum_{a=1}^6 F_a} = \frac{1}{3}, \quad a = 1, \dots, 6 \quad (\text{A.16})$$

Notice that the $\text{SL}(4, \mathbb{Z})$ basis for the Reeb vector in the toric data method is in general not connected to the analogous basis used in the Molien formula approach.

B Three dimensional crystals and “dual ABJM theory”

For brane tilings, two of the three complex dimensions of the Calabi-Yau moduli space are related to the two non-trivial cycles of the two-torus (and there is also a “radial” direction). It is therefore natural to associate three dimensional crystals to Calabi-Yau fourfolds. A simple three dimensional generalization of the hexagonal tiling gives the brane crystal for \mathbb{C}^4 [27].

Unlike in the case of tilings, in three dimensions it is *a priori* unclear how one should determine the number of gauge groups. The edges correspond to matter multiplets but their charges cannot be easily read off from the crystal. The order of the fields in the superpotential is also ambiguous.

This latter ambiguity can be fixed by choosing an oriented plane at each vertex. When glued together, these planes form a ribbon graph which is on top of the brane crystal. The simplest example is depicted in Figure 21 (i) where the ribbon graph is shown in blue. Once we have fixed the local plane at the vertices by specifying how the ribbon graph is spanned on the brane crystal, we can turn to the problem of gauge groups.

A natural choice is to associate the gauge groups to paths on the crystal edges which always turn left (or always turn right). In two dimensions, this simply gives the faces of the brane tiling. In three dimensions, however, such paths do not close. They are in fact infinite spirals as the reader may check in Figure 21. Each of them defines a “Burgers vector”: by moving around a would-be face once, we find ourselves in a different fundamental domain. The “faces” spanned by these spirals are infinite half-helicoids. Note that the Burgers vectors for the faces are parallel (opposite direction).

We see that this definition results in two distinct gauge groups that correspond to the darker blue and lighter blue areas of the ribbon graph in Figure 21 (i). The matter multiplets are charged under the gauge groups that correspond to the two sides of the

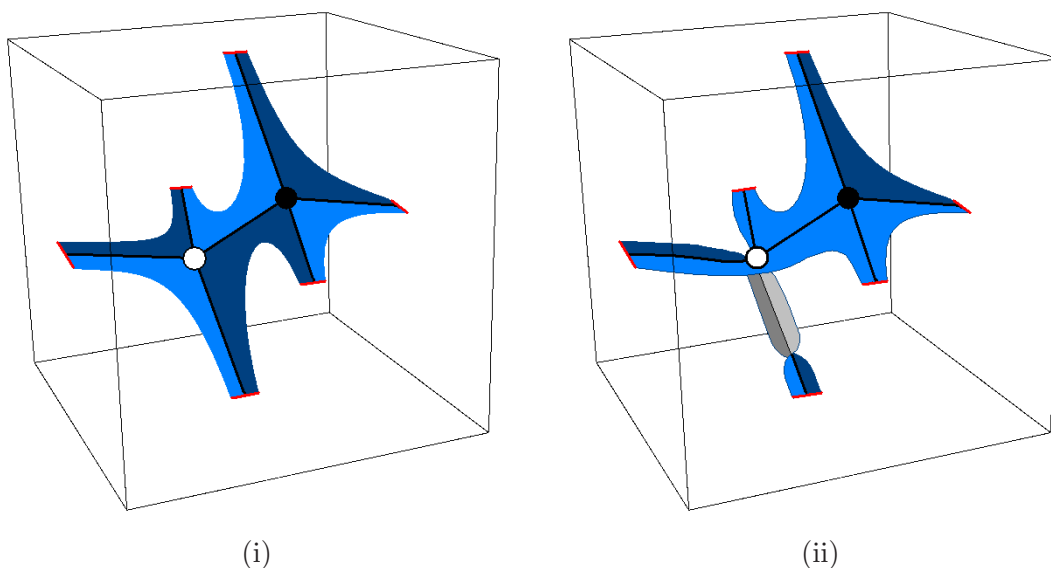


Figure 21. Brane crystal for \mathbb{C}^4 with ribbon graphs: (i) this choice gives the ABJM theory. The red lines indicate where the ribbon touches the wall of the fundamental domain. (ii) The local plane at the white vertex is now different. Gray color indicates the other side of the ribbon. This choice gives the “dual ABJM theory”.

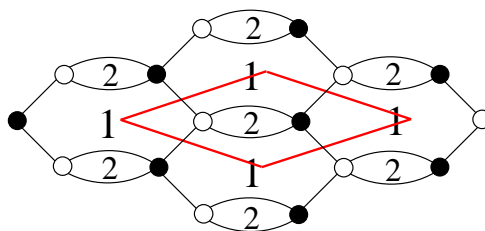


Figure 22. Brane tiling for the “dual ABJM theory”.

ribbon near the particular edge. With this definition, we end up with the ABJM theory with four bifundamental fields between the two gauge groups.

From a certain direction, this three dimensional zinc-blende crystal looks like a square lattice Figure 1. This is the origin of the similarity of ABJM to the conifold theory.

So far we have been discussing a particular choice of ribbon graph on the crystal. Another choice is shown in Figure 21 (ii). This choice again gives two gauge groups. There are two bifundamental fields between them and one of the groups has two adjoint fields as well. The interactions are specified by the superpotential

$$W = [\phi_1, \phi_2]AB \tag{B.1}$$

We can call this model the “dual ABJM theory”. An equivalent two-dimensional tiling is shown in Figure 22. In 3+1 dimensions, this tiling would give an inconsistent theory: some fields would have vanishing R-charges. In 2+1 dimensions, however, we cannot exclude such models from the discussion on similar grounds. Such “inconsistent” tilings would then greatly enlarge the set of models describable by two-dimensional tilings.

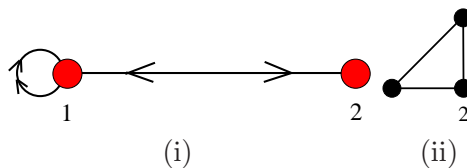


Figure 23. (i) Quiver corresponding to the “dual ABJM theory”. (ii) Toric diagram: one of the external nodes is a double point.

The Kasteleyn matrix for the “dual ABJM theory” is

$$K = 1 + x + y + z \tag{B.2}$$

The moduli space at level k is therefore $\mathbb{C}^2/\mathbb{Z}_k \times \mathbb{C}^2$. At level one, this is simply \mathbb{C}^4 . Thus, we expect the theory to be dual to ABJM in the sense of Seiberg-duality. This needs further investigation.

In general, one can introduce “Dehn twists” for the ribbon graph along the crystal edges. The possibilities are constrained by the fact that the graph must be oriented otherwise the ordering of the fields in the superpotential is again ambiguous. Note that there is no clear difference between the “tiling” and the “untwisted tiling” (a.k.a. shiver) in three dimensions. For \mathbb{C}^4 , using the high symmetry of the crystal, one can fix the local plane at the black vertex as in Figure 21 without losing generality. There are then six choices for the local plane at the white vertex (some of them are mirror images to the ABJM and dual ABJM ribbon-crystals).

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